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Modules with reduced endomorphism rings

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In this paper, we study endo-reduced modules as modules whose endomorphism rings have no nonzero nilpotent elements. We characterize their properties for different classes of modules, including $\mathcal{K}$-non-singular modules, multiplication modules and finitely generated modules over commutative Dedekind domains. In the subcategory of finitely generated modules, it is shown that the class of rings R for which every faithful multiplication R-module is endo-reduced is precisely that of reduced rings; while the class of rings R for which every multiplication R-module is endo-reduced is precisely that of von Neumann regular rings. Characterizations of when an endo-reduced module will be a reduced module are given. We prove that a finitely generated module over a principal ideal domain (PID) is endo-reduced exactly if it is either a semisimple module with pair-wise non-isomorphic submodules or a torsion-free module which is isomorphic to the underlying ring.

Keywords: Endo-reduced module; reduced module; endomorphism ring; reduced ring; von Neumann regular ring.

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1. Introduction

We study modules whose endomorphism rings have no nonzero nilpotent endomorphisms. We call a ring R *reduced* if it has no nonzero nilpotent elements. It is *Abelian* if all its idempotents are central. The study of reduced rings has been an important topic in Ring Theory because this notion helps in the description of structures of rings and modules related to them. Reduced rings and Abelian rings have been extensively studied and extended by a number of authors [3–6, 13, 17, 20]. The notion of an Abelian ring was extended to an endabelian (or Abelian in [19, Definition 4.4]) module as a module theoretic analogue by Călugăreanu and Schultz [6] via the endomorphism ring of the module. An R-module M is said to be an *endabelian module* (or Abelian module) if its endomorphism ring is Abelian. In 2004, the notion of a reduced ring was extended to a reduced module as a module theoretic analogue by Lee and Zhou [16]. An R-module M is *reduced* if for each $m \in M$ and $a \in R$, $ma^2 = 0$ implies $mRa = 0$. Even though a general definition for a reduced module was given and several properties of a reduced ring were generalized to a reduced module, not much work was done by utilizing the endomorphism rings of modules (see related literature on reduced modules in [8, 9, 11, 16]).

Endomorphism rings of modules are important due the role they play in Morita contexts. In this paper, our focus of the study is on modules whose endomorphism rings are reduced. We call them *endo-reduced modules*. We characterize an endo-reduced module in different classes of modules. On top of the natural desire to extend the notion of a reduced ring to a general module theoretic setting, we investigate the following questions: (1) Does the endo-reduced module property extend to the direct summands and direct sums of modules? (2) How far apart can the (Lee and Zhou) reduced module and the endo-reduced module properties be in a module theoretic setting? (3) Which classes of rings R are characterized by the endo-reduced modules? These questions motivate our study and various answers are provided while using simple but efficient techniques.

This paper is organized as follows. In Sec. 2, we introduce the notion of endo-reduced modules. We delve into its properties and establish certain characterizations for endo-reduced modules within distinct classes of modules. We determine the necessary and sufficient condition for arbitrary direct sums of endo-reduced modules to be endo-reduced (Theorem 1). After showing that endo-reduced modules do not possess the common forms of non-singularity, we prove that the uniform $\mathcal{K}$-non-singular modules are endo-reduced modules (Theorem 2). This extends the result by Johnson–Wong in [10, Theorem 1.7] to the class of $\mathcal{K}$-non-singular modules. We examine scenarios wherein endo-reduced modules can be described through the kernels or images of their endomorphisms. As a consequence, we prove the case when, for P-injective or P-flat modules the notions of "endo-reduced module" and "reduced module" are indistinguishable (Theorems 3 and 4).

In Sec. 3, our focus is on the notion of endo-reduced module in multiplication modules. In the subcategory of finitely generated modules, we establish the following: (1) The endo-reduced multiplication modules coincide with reduced multiplication modules (Proposition 12); (2) The rings R for which every faithful multiplication R-module is an endo-reduced module are the reduced rings (Theorem 5); (3) The class of rings R for which every multiplication R-module is endo-reduced is precisely that of the von Neumann regular rings (Theorem 6). Lastly, the investigations in Sec. 4 are on the question of when an arbitrary direct sum M of cyclic submodules over a commutative Dedekind domain is endo-reduced. We prove that this holds if M is either a semisimple module with pair-wise non-isomorphic submodules or M is a torsion-free module which is isomorphic to the underlying Dedekind domain (Theorem 7). Hence a finitely generated Abelian group G is endo-reduced as a $\mathbb{Z}$-module if and only if G is isomorphic $\bigoplus_{p_\alpha \in \mathcal{P}} \mathbb{Z}/p_\alpha\mathbb{Z}$, where $\mathcal{P}$ is a collection of distinct prime numbers p_α of $\mathbb{Z}$, or G is isomorphic to $\mathbb{Z}_\mathbb{Z}$.

Throughout this paper, R is a ring with unity and M is a unital right R-module. For a right R-module M, let $S = \text{End}_R(M)$ denote the endomorphism ring of M. Then M can be viewed as a left S-right R-bimodule. The notations $N \subseteq M, N \subseteq^{\text{ess}} M$ and $N \subseteq^{\oplus} M$ represent: N is a submodule, an essential submodule and a direct summand of M, respectively. By $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ and $\mathbb{N}$ we denote the set of real, rational, integer and natural numbers, respectively. For $1 < n \in \mathbb{N}, \mathbb{Z}_n$ denotes the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}_{p^\infty}$ denotes the Prüfer p-group where p is a prime. We use $M^{(\mathcal{J})}$ ($\mathcal{J}$ an index set) to denote the direct sum of $|\mathcal{J}|$ copies of M. By $\mathcal{R}ad(M)$, we denote the Jacobson radical of M (the intersection of its maximal submodules). We also denote $\text{Ann}^r_M(X) = \{m \in M : Xm = 0\}, \text{Ann}^r_S(X) = \{\varphi \in S : X\varphi = 0\}$, and $\text{Ann}^l_S(X) = \{\varphi \in S : \varphi X = 0\}$, for $\emptyset \neq X \subseteq S; \text{Ann}^l_S(N) = \{\varphi \in S : \varphi N = 0\}, \text{Ann}_R(N) = \{r \in R : Nr = 0\}$ and $(N : M) = \{x \in R \mid Mx \subseteq N\}$ for $N \subseteq M; \text{Ann}^r_R(I) = \{a \in R : Ia = 0\}$ and $\text{Ann}^l_R(I) = \{a \in R : aI = 0\}$ for $\emptyset \neq I \subseteq R$. For $\varphi \in S, \text{Ann}^r_M(\varphi) = \ker(\varphi)$ and $\text{Im}(\varphi)$ stand for the kernel and the image of φ, respectively.

We conclude this section with the following (Propositions 1–3) which are easy consequences of well-known facts and results (see for instance: [11, Lemma 1.5 and Corollary 1.6; 20, Lemmas 5.1 and 12.2]). They will be inherently useful to our study.

Proposition 1. *The following statements are equivalent for an R-module M:*

(1) *M_R is a reduced module*;
(2) *For each $a \in R$, $Ann^l_M(a) \cap Ma = 0$ and if $ma = 0$, then $mra = 0$ for all $r \in R$ and $m \in M$.*

By exploiting the connection between an R-module M and its endomorphisms ring $S = \text{End}_R(M)$, the following gives a characterization of reduced S-modules.

Proposition 2. *The following statements are equivalent for an R-module M and $S = End_R(M)$:*

(1) *${}_SM$ is a reduced module*;

(2) *For each* $\varphi \in S, Ann^r_M(\varphi) = Ann^r_M(\varphi^2)$ *and if* $\varphi m = 0$, *then* $\varphi\phi m = 0$ *for every* $\phi \in S$ *and* $m \in M$;
(3) *For each* $\varphi \in S, Ann^r_M(\varphi) \cap \varphi M = 0$ *and if* $\varphi\phi m = 0$, *then* $\phi\varphi m = 0$ *for every* $\phi \in S$ *and* $m \in M$.

Proposition 3. *The following statements hold true for a ring* R:

(1) R *is a reduced ring if and only if* $Ann^l_R(a^2) = Ann^l_R(a)$ *if and only if* $Ann^r_R(a^2) = Ann^r_R(a)$ *for every* $a \in R$;
(2) *If* R *is a reduced ring, then* $ab = 0$ *implies* $ba = 0$ *for every* $a, b \in R$;
(3) *If* R *is a reduced ring, then every idempotent element of* R *is central in* R.

2. Endo-Reduced Modules

Definition 2.1. Let R be a ring and M a right R-module. Then M is an *endo-reduced module* if $\mathrm{End}_R(M)$ is a reduced ring. Equivalently, for all $\varphi \in \mathrm{End}_R(M), \varphi^2 = 0$ implies that $\varphi = 0$.

Examples of endo-reduced modules are given below. We call a ring R *von Neumann* (respectively, *strongly*) *regular* if for each $a \in R$ there exists an $x \in R$ such that $a = axa$ (respectively, $a = xa^2$).

Example 2.1. (a) Domains, subrings of reduced rings, and all (sub) direct products of reduced rings are reduced as rings. Therefore, every module whose endomorphism ring is a subdirect sum (respectively, product) of domains (respectively, reduced rings) is endo-reduced. Thus by the well-known Schur's lemma, that the endomorphism ring of a simple module is a division ring, every simple module is endo-reduced. But the converse to this is not true in general. For instance: The $\mathbb{Z}$-module $\mathbb{Z}$ is endo-reduced (note that $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}) = \mathbb{Z}$ is just a domain), and so is the submodule $4\mathbb{Z}$ of $\mathbb{Z}$. However, neither $\mathbb{Z}$ nor $4\mathbb{Z}$ is simple. The $\mathbb{Z}$-module $\mathbb{Z}_p$, where p is a prime integer, is endo-reduced since it is simple.

(b) Every module M whose endomorphism ring $\mathrm{End}_R(M)$ is a strongly regular ring is an endo-reduced module. To see this, let $\varphi \in \mathrm{End}_R(M)$ such that $\varphi^2 = 0$. Since $\mathrm{End}_R(M)$ is strongly regular, there exists $\phi \in \mathrm{End}_R(M)$ such that $\varphi\phi\varphi = \varphi$ and $\varphi\phi = \phi\varphi$. This gives a central idempotent $\varphi\phi$. Hence $(\varphi\phi)M = (\varphi\phi)^2M = (\phi\varphi)^2M = \varphi^2\phi^2M \subseteq \varphi^2M$. So $\varphi\phi = 0$, and hence $\varphi = \varphi\phi\varphi = 0$.

(c) Consider $\mathbb{Z}$ the ring of integers and the Prüfer p-group $\mathbb{Z}_{p^\infty}$. It is well known that $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})$ is the ring of p-adic integers. Since the ring of p-adic integers is a commutative domain, $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})$ is a reduced ring. Hence $\mathbb{Z}_{p^\infty}$ is an endo-reduced $\mathbb{Z}$-module by (a). Note that $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})$ is neither a division ring nor a strongly regular ring.

(d) Every R-module M_R which is reduced as an S-module, where $S = \mathrm{End}_R(M)$, is an endo-reduced module. Let $\varphi \in S$ such that $\varphi^2M = 0$. Then $\varphi^2m = 0$

for each $m \in M$. As ${}_SM$ is a reduced module, $\varphi Sm = 0$. It follows that $0 = \varphi 1_M m = \varphi m$ for each $m \in M$. Hence $\varphi = 0$, and M is an endo-reduced module.

Let R be a ring and M a right R-module with $S = \mathrm{End}_R(M)$. While the notions of a "reduced R-module" and a "reduced S-module" coincide with the definition of an "endo-reduced module" when the module $M = R_R$, the next examples show the independence of our notion from the former two notions for the case of arbitrary modules. The examples show that while the class of endo-reduced module properly contains the class of reduced S-modules, it is independent from the class of reduced R-modules. The first implication follows from Example 2.1(d). Example 2.2 exhibits endo-reduced modules which are neither reduced S-modules nor reduced R-modules, while Example 2.3 shows reduced R-modules which are not endo-reduced. In general,

$$\begin{array}{ccccc} {}_SM \text{ is a reduced module} & \Rightarrow & M_R \text{ is an endo-reduced} & \not\Rightarrow & M_R \text{ is a reduced} \\ \text{module} & & \text{module} & & \text{module.} \\ & \not\Leftarrow & & \not\Leftarrow & \end{array}$$

Example 2.2. Let $R = \mathbb{Z}$ be a ring and consider the Prüfer p-group $M = \mathbb{Z}_{p^\infty}$ and $S = \mathrm{End}_R(M)$. By Example 2.1(c), M is an endo-reduced module. However, neither ${}_SM$ nor M_R is a reduced module. Otherwise, for the surjective endomorphism $\varphi : M \to M$ defined by $\varphi m = mp$ for every $m \in M$, Proposition 2 would imply $\mathrm{Ann}^r_M(\varphi) \cap \varphi M (= \mathrm{Ann}^r_M(\varphi) \cap M = \mathrm{Ann}^r_M(\varphi) \subseteq^{\mathrm{ess}} M)$ is a trivial submodule of M; which is a contradiction. So ${}_SM$ is not a reduced module. Analogously, using Proposition 1, M_R is not a reduced module as well since $\mathrm{Ann}^l_M(p) \cap Mp = \mathrm{Ann}^r_M(\varphi) \cap \varphi M \neq 0$.

Example 2.3. (a) Every direct product of reduced R-modules is a reduced R-module. In particular, the free $\mathbb{Z}$-module $\mathbb{Z}^{(\mathcal{J})}$, for any index set $|\mathcal{J}| \neq 1$, is a reduced $\mathbb{Z}$-module which is not an endo-reduced module (see Theorem 1 and Corollary 1). Hence $\mathbb{Z}[x] (\cong \mathbb{Z}^{(\mathbb{N})})$ is a reduced $\mathbb{Z}$-module which is not an endo-reduced $\mathbb{Z}$-module.

(b) Let R be a commutative von Neumann regular ring with a non-finitely generated maximal ideal $\mathcal{M}$. Consider $\overline{R} := R/\mathcal{M}$ and $M := R \oplus \overline{R}$. Then M is a reduced R-module by [11, Example 5.4]. However, M is not an endo-reduced module. Note that $\mathrm{Hom}_R(\overline{R}, R) = 0$ and

$$\mathrm{End}_R(M) \cong \begin{pmatrix} \mathrm{End}_R(R) & \mathrm{Hom}_R(\overline{R}, R) \\ \mathrm{Hom}_R(R, \overline{R}) & \mathrm{End}_R(\overline{R}) \end{pmatrix} \cong \begin{pmatrix} R & 0 \\ \overline{R} & \overline{R} \end{pmatrix},$$

contains a nonzero endomorphism φ corresponding to $\left(\begin{smallmatrix} 0 & 0 \\ \bar{1} & \bar{0} \end{smallmatrix}\right)$ for which $\varphi^2 = 0$.

Proposition 4 will be used without mention throughout the paper. It characterizes endo-reduced modules in terms of right (left) annihilators of any finitely generated right (left) ideals of $\mathrm{End}_R(M)$.

Proposition 4. *The following conditions are equivalent for a module M and $S = End_R(M)$:*

(1) *M is an endo-reduced module;*
(2) *$Ann^r_S(\varphi^2) = Ann^r_S(\varphi)$ (respectively, $Ann^l_S(\varphi^2) = Ann^l_S(\varphi)$) for every $\varphi \in S$;*
(3) *$Ann^r_S(\varphi) \cap \varphi S = 0$ (respectively, $Ann^l_S(\varphi) \cap S\varphi = 0$) for every $\varphi \in S$;*
(4) *$Ann^r_S(I) \cap I = 0$ (respectively, $Ann^l_S(I) \cap I = 0$) for every right (respectively, left) ideal I of S;*
(5) *$Ann^r_S(I) \cap I = 0$ (respectively, $Ann^l_S(I) \cap I = 0$) for every finitely generated right (respectively, left) ideal $I = \langle \varphi_1, \ldots, \varphi_n \rangle$ of S.*

Proof. (1)$\Leftrightarrow$(2) Follows from Proposition 3 and the fact that S is a reduced ring.

(2)$\Rightarrow$(3) Let $\psi \in \mathrm{Ann}^r_S(\varphi) \cap \varphi S$. Then $\psi = \varphi\phi$ for some $\phi \in S$ and $\varphi\psi = \varphi^2\phi = 0$. Hence $\phi \in \mathrm{Ann}^r_S(\varphi^2) = \mathrm{Ann}^r_S(\varphi)$, i.e. $\psi = \varphi\phi = 0$. The proof for $\mathrm{Ann}^l_S(\varphi) \cap S\varphi = 0$ is analogous.

(3)$\Rightarrow$(4) Let I be a right ideal I of S and chose $\psi \in \mathrm{Ann}^r_S(I) \cap I$. Then $\psi \in \bigcap_{\varphi \in I} \mathrm{Ann}^r_S(\varphi)$ and $\psi \in I$. Since $\psi \in I$ implies $\psi = \psi \cdot 1 \in \psi S$, it follows that $\psi \in \mathrm{Ann}^r_S(\psi) \cap \psi S = 0$ by (3). The proof of $\mathrm{Ann}^l_S(I) \cap I = 0$ follows in a similar manner.

(4)$\Rightarrow$(5) This is straightforward.

(5)$\Rightarrow$(1) Since for any $\varphi \in S, \varphi S$ is a finitely generated right ideal with one generator, $\varphi^2 = 0$ implies that $\varphi \in \mathrm{Ann}^r_S(\varphi) \cap \varphi S = \mathrm{Ann}^r_S(\varphi S) \cap \varphi S = 0 \Rightarrow \varphi = 0$. Therefore, M is an endo-reduced module. □

Remark 1. Let R be a ring and M_R be a nontrivial R-module.

(a) If R is commutative and M_R is an endo-reduced module, then $R/\mathrm{Ann}_R(M)$ is a reduced ring (Example 2.1(a),(d)). Note that $R/\mathrm{Ann}_R(M)$ is isomorphic to a subring of a reduced ring $\mathrm{End}_R(M)$.
(b) For an element $a \in R$, we define $\varphi_a : R \to R$ by $\varphi_a x := ax$, and we put $\mathrm{Ann}^r_R(a) = \mathrm{Ann}^r_R(\varphi_a)$. Using Propositions 3 and 4, R is a reduced ring $\Leftrightarrow$ $\mathrm{Ann}^r_R(I) \cap I = 0 \Leftrightarrow \mathrm{Ann}^l_R(I) \cap I = 0$ for every (cyclic) left (right) ideal I of R.
(c) If M is an endo-reduced module (in view of Proposition 3(2) and Definition 2.1), then $\varphi\phi M = 0$ implies $\phi\varphi M = 0$ for every $\varphi, \phi \in \mathrm{End}_R(M)$.

Much as $\mathbb{Z}_\mathbb{Z}$ and its submodule $4\mathbb{Z}_\mathbb{Z}$ are endo-reduced modules (both $\mathrm{End}_\mathbb{Z}(\mathbb{Z})$ and $\mathrm{End}_\mathbb{Z}(4\mathbb{Z})$ are domains), the endo-reduced module property does not always transfer from a module to each of its submodules or conversely as the next Example 2.4 exhibits. The example also shows that epimorphic images of endo-reduced modules are not necessarily endo-reduced.

Example 2.4. (a) $\mathbb{Z}_4$ is not an endo-reduced $\mathbb{Z}$-module. To see this, define $\varphi \in \mathrm{End}_\mathbb{Z}(\mathbb{Z}_4)$ by $\varphi\overline{1} = \overline{2}$. Then $\varphi^2\overline{1} = \overline{0}$ and so, $\overline{1} \in \mathrm{Ann}^r_S(\varphi^2)$. Since $\overline{1} \notin \mathrm{Ann}^r_S(\varphi), \mathbb{Z}_4$ is not endo-reduced by Proposition 4. However, the submodule $2\mathbb{Z}_4$ of $\mathbb{Z}_4$ is an endo-reduced $\mathbb{Z}$-module because $2\mathbb{Z}_4 \cong_\mathbb{Z} \mathbb{Z}_2$ (a simple module).

In general, even though $\mathbb{Z}_{p^2}$ (p a prime) is not an endo-reduced $\mathbb{Z}$-module, its submodule $p\mathbb{Z}_{p^2}$ is an endo-reduced module because $p\mathbb{Z}_{p^2} \cong \mathbb{Z}_p$ is simple.

(b) Submodules of an endo-reduced module are not endo-reduced modules in general. In the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}_2$, both $\mathbb{Q}$ and $\mathbb{Z}_2$ are endo-reduced $\mathbb{Z}$-modules and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_2) = 0 = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, \mathbb{Q})$. So $\mathrm{End}_R(M) = \mathrm{End}_R(\mathbb{Q} \oplus \mathbb{Z}_2) \cong \mathrm{End}_R(\mathbb{Q}) \oplus \mathrm{End}_R(\mathbb{Z}_2)$ is a subdirect product of is division rings. Thus M is endo-reduced by Example 2.1(a). However, since the nonzero $\mathbb{Z}$-endomorphism $\varphi : \mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z} \oplus \mathbb{Z}_2$, defined by $\varphi(m, \overline{n}) = (0, \overline{m})$, gives $\varphi^2(\mathbb{Z} \oplus \mathbb{Z}_2) = 0$, the submodule $\mathbb{Z} \oplus \mathbb{Z}_2$ of M is not an endo-reduced $\mathbb{Z}$-module.

(c) In connection to (a) and (b), the Prüfer p-group $\mathbb{Z}_{p^\infty}$ is a (quasi-)injective endo-reduced $\mathbb{Z}$-module by Example 2.2, while its submodule $\mathbb{Z}_{p^2}$ is not an endo-reduced module by (a).

(c) $\mathbb{Z}_4$ is not endo-reduced as a $\mathbb{Z}$-module by (a), but it is the image of a (quasi-) projective module $\mathbb{Z}_{\mathbb{Z}}$ over $4\mathbb{Z}$ which are both endo-reduced modules.

On the other hand, a direct summand of an endo-reduced module inherits the property.

Proposition 5. *Every direct summand of an endo-reduced module M is endo-reduced.*

Proof. Let M be an endo-reduced module, N be a direct summand of M, i.e. $N = eM$ for some $e^2 = e \in \mathrm{End}_R(M)$ and $\varphi \in \mathrm{End}_R(N)$ such that $\varphi^2 N = 0$. Then $(\varphi e)^2 M = \varphi^2 e^2 M = \varphi^2 eM = \varphi^2 N = 0$ because, by Proposition 3, e is central in $\mathrm{End}_R(M)$. Since M is endo-reduced and $\varphi e \in \mathrm{End}_R(M), \varphi e = 0$, and hence $\varphi N = \varphi eM = 0$. Therefore, $\varphi = 0$, i.e. N is endo-reduced. □

Remark 2. The class of reduced R-modules is closed under direct products, submodules and therefore direct sums.

Example 2.5 shows that the converse of Proposition 5 is not true in general for endo-reduced modules. While a direct summand of an endo-reduced module is endo-reduced, the example further illustrates that the endo-reduced module property does not go to direct sums of endo-reduced modules.

Example 2.5. (a) Simple modules are endo-reduced but a direct sum of simple modules (semisimple modules) may not be endo-reduced. Let $M := \mathbb{R} \oplus \mathbb{R}$, a direct sum of simple R-modules. Then we have $\mathrm{End}_{\mathbb{R}}(M) = \mathrm{Mat}_2(\mathbb{R})$, which is not a reduced ring but just a von Neumann regular ring. In particular, since $\varphi^2 M = 0$ for the element $0 \neq \varphi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}_R(M)$, M is not endo-reduced. In fact, a semisimple ring need not be reduced.

(b) Consider the ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and the idempotent $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $M := R_R$ and consider $0 \neq \varphi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{End}_R(M) \cong R$. Then $\varphi^2 M = 0$, and thus M is not endo-reduced. However, for $eR = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$, we have $\mathrm{End}_R(eR) \cong \mathbb{Z}$ showing that eR is endo-reduced. Furthermore, $(1-e)R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{pmatrix} \cong \mathrm{End}_R((1-e)R) \cong \mathbb{Z}$. So both eR and $(1-e)R$ are endo-reduced R-modules, while their direct sum $M_R \cong eR \oplus (1-e)R$ is not so.

We call an R-module M *Abelian* if $\mathrm{End}_R(M)$ is an Abelian ring. It is *Dedekind finite* or *directly finite* if whenever N is a submodule of M such that M is isomorphic to the module $M \oplus N$, then $N = 0$. Endo-reduced modules form a proper subclass of modules that are Abelian or directly finite. In particular, not every Abelian or directly finite module is endo-reduced.

Lemma 1. *Let* $\{M_\alpha \,|\, \alpha \in \mathcal{J}\}$ *be a family of R-modules such that* $M = \bigoplus_{\alpha \in \mathcal{J}} M_\alpha$ *(*$\mathcal{J}$ *an index set). If* M *is an endo-reduced module, then*

(1) *Hom*$(M_\alpha, M_\beta) = 0$ *for all* $\alpha \neq \beta$, *where* $\alpha, \beta \in \mathcal{J}$;
(2) *End*$_R(M) =$ *End*$_R(\bigoplus_{\alpha \in \mathcal{J}} M_\alpha) \cong \prod_{\alpha \in \mathcal{J}}$ *End*$_R(M_\alpha)$;
(3) M *is Abelian*;
(4) M *is Dedekind finite.*

Proof. (1) Assume M is endo-reduced. To show that $\mathrm{Hom}_R(M_\alpha, M_\beta) = 0$ for every $\alpha \neq \beta$, pick $\varphi \in \mathrm{Hom}_R(M_\alpha, M_\beta)$ and assume $\varphi \neq 0$. Let $e_\alpha : M \to M_\alpha$ denote the canonical projection and let $\iota_\beta : M_\beta \to M$ denote inclusion map. Since $\varphi \neq 0, \phi = \iota_\beta \varphi e_\alpha$ is nonzero. It follows that for each $(m_\alpha, m_\beta) \in M_\alpha \oplus M_\beta$, $\phi^2(m_\alpha, m_\beta) = \iota_\beta \varphi e_\alpha \iota_\beta \varphi e_\alpha (m_\alpha, m_\beta) = \iota_\beta \varphi e_\alpha \iota_\beta \varphi m_\alpha = \iota_\beta \varphi e_\alpha (0, \varphi m_\alpha) = \iota_\beta \varphi 0 = 0$. So ϕ is a nonzero endomorphism of M for which $\phi^2 M = 0$, a contradiction.

(2) This follows from the fact in (1) that $\mathrm{Hom}(M_\alpha, M_\beta) = 0$ for all $\alpha \neq \beta$, where $\alpha, \beta \in \mathcal{J}$.

(3) Since $\mathrm{End}_R(M)$ is a reduced ring, every idempotent endomorphism is central by Proposition 3. It follows that $\mathrm{End}_R(M)$ is an Abelian ring and so is M by definition.

(4) Let N be a submodule of M such that M is isomorphic to the module $M \oplus N$. Since $\mathrm{Hom}_R(M, N) = 0 = \mathrm{Hom}_R(N, M)$ by (1), $N = 0$. This proves that M is Dedekind finite. □

The converse of each of the statements (1)–(4) in Lemma 1 is false. Note that the $\mathbb{Z}$-module $\mathbb{Z}_4$ is Abelian (and so Dedekind finite) but $\mathbb{Z}_4$ is not an endo-reduced $\mathbb{Z}$-module by Example 2.4.

Motivated by the question of which classes of rings are characterized by the endo-reduced modules, we have the following. More explicit characterizations of rings are given in Sec. 3.

Proposition 6. *Consider the following conditions*:

(1) *Every finitely generated* (*respectively*, *every* 2*-generated*) *R-module is an endo-reduced module*;
(2) *R is a division ring.*

Then (1) $\Rightarrow$ (2). *If we restrict the modules in* (1) *to be also indecomposable nontrivial modules*, *then the converse holds.*

Proof. (1)$\Rightarrow$(2) Let a be a nonzero element in R and let $\gamma : R \to R/aR$ be the coset map. Since $M = R \oplus (R/aR)$ is endo-reduced by hypothesis, γ is a zero endomorphism of M by Lemma 1. In particular, $aR = R$. Since a is a unit, R is a division ring.

Now suppose M is an indecomposable module.

(2)$\Rightarrow$(1) Since M is finitely generated and R is a division ring, $M \cong R^{\mathcal{J}}$ for some index $\mathcal{J}$ [21, Sec. 20.10]. As M is indecomposable, $M \cong R_R$, which is an endo-reduced module. □

The implication (2) $\Rightarrow$ (1) in Proposition 6 may not be true if we do not impose the indecomposability property. For the $\mathbb{R}$-vector space $M = \mathbb{R} \oplus \mathbb{R}, \mathrm{End}_{\mathbb{R}}(M)$ is a von Neumann regular ring which is not reduced (Example 2.5). We improve Proposition 6 with Proposition 7. Call module M a Fitting module if for all $\varphi \in \mathrm{End}_R(M), M = \ker(\varphi^n) \oplus \mathrm{Im}(\varphi^n)$ for some $n \in \mathbb{N}$.

Lemma 2. *The following statements are equivalent for a Fitting module* M:

(1) *M is an indecomposable endo-reduced module*;
(2) *Every nonzero endomorphism $\varphi \in End_R(M)$ is an epimorphism*;
(3) *Every nonzero endomorphism $\varphi \in End_R(M)$ is a monomorphism*;
(4) *$End_R(M)$ is a division ring.*

Proof. (2)$\Leftrightarrow$(3)$\Leftrightarrow$(4)$\Rightarrow$(1) are obvious.

(1)$\Rightarrow$(4) Let $\varphi \in \mathrm{End}_R(M)$ be nonzero. As M is indecomposable and Fitting, $\ker(\varphi) = 0$ and $\varphi M = M$. So φ is an automorphism of M. Thus $\mathrm{End}_R(M)$ is a division ring. □

Proposition 7. *Let R be an Artinian ring and let M be a nontrivial finitely generated R-module. Then the following statements are equivalent*:

(1) *M is both indecomposable and endo-reduced*;
(2) *$End_R(M)$ is a division ring.*

Proof. (1)$\Rightarrow$(2) In view of [21, Sec. 31.13], the information in the hypothesis implies that M is a Fitting module. Now given (1), $\mathrm{End}_R(M)$ is a division ring by Lemma 2.

(2)$\Rightarrow$(1) This is obvious. □

In Theorem 1, we provide the necessary and sufficient condition for arbitrary direct sums of endo-reduced modules to be endo-reduced.

Theorem 1. *Let* $\{M_\alpha \mid \alpha \in \mathcal{J}\}$ *be a family of R-modules such that* $M = \bigoplus_{\alpha\in\mathcal{J}} M_\alpha$ ($\mathcal{J}$ *an index set). Then M is an endo-reduced module if and only if* M_α *is endo-reduced for each* $\alpha \in \mathcal{J}$ *and* $Hom(M_\alpha, M_\beta) = 0$ *for all* $\alpha \neq \beta$, *where* $\alpha, \beta \in \mathcal{J}$.

Proof. The necessity is clear by Proposition 5 and Lemma 1. To prove sufficiency, we assume that M_α is endo-reduced for all $\alpha \in \mathcal{J}$. Take $\varphi \in \mathrm{End}(M)$ such that $\varphi^2 M = 0$. Since $\mathrm{Hom}(M_\alpha, M_\beta) = 0$ for all $\alpha \neq \beta$, where $\alpha, \beta \in \mathcal{J}$, we have $\mathrm{End}(M) = \mathrm{End}_R(\bigoplus_{\alpha\in\mathcal{J}} M_\alpha) \cong \prod_{i\in\mathcal{J}} \mathrm{End}(M_\alpha)$ and $\varphi^2 M_\alpha \subseteq \varphi M_\alpha \subseteq M_\alpha$ for all $\alpha \in \mathcal{J}$. Hence we can decompose $\varphi := \bigoplus_{\alpha\in\mathcal{J}} \varphi_\alpha$ and $\varphi^2 := \bigoplus_{\alpha\in\mathcal{J}} \varphi_\alpha^2$ as finitely nonzero sums, where $\varphi_\alpha \in \mathrm{End}(M_\alpha)$. It follows that $0 = \varphi^2 M = \bigoplus_{\alpha\in\mathcal{J}} \varphi_\alpha^2 M_\alpha$, and so $\varphi_\alpha^2 M_\alpha = 0$ for all $\alpha \in \mathcal{J}$. Since each M_α is endo-reduced, $\varphi_\alpha = 0$ for all $\alpha \in \mathcal{J}$. Thus $\varphi := \bigoplus_{\alpha\in\mathcal{J}} \varphi_\alpha = 0$. □

Remark 3. (a) While Example 2.4(c) shows that homomorphic images of endo-reduced modules do not inherit this property, using Theorem 1 we can show that the quotient $\mathbb{Q}/\mathbb{Z}$ of endo-reduced $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Z}$ is an endo-reduced $\mathbb{Z}$-module. In fact, $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p\in\mathcal{P}} \mathbb{Z}_{p^\infty}$ where $\mathcal{P}$ is a collection of prime numbers and $\mathbb{Z}_{q^\infty}$ is the p-primary component of $\mathbb{Q}/\mathbb{Z}$. We have that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, Z_{q^\infty}) = 0$ for every $p \neq q$.

(b) Let M be an R-module with a von Neumann regular endomorphism ring. If M is an endo-reduced module, then so are $\ker(\varphi)$ and $\mathrm{Im}(\varphi)$ for every $\varphi \in \mathrm{End}_R(M)$ by Theorem 1 and [15, Remark 2.8].

Corollary 1. *Let* $M = \bigoplus_{\alpha\in\mathcal{J}} N_\alpha$ *be a direct sum of copies of a nontrivial R-module* N_α, *where* $N_\alpha = N$ *and* $\mathcal{J} \neq \emptyset$ *is an arbitrary index set. If M is endo-reduced, then* $|\mathcal{J}| = 1$.

Proof. This is a consequence of Lemma 1 and Theorem 1. □

Corollary 2. *Let* $\{N_\alpha \mid \alpha \in \mathcal{J}\}$ *be a family of simple modules which are non-isomorphic in pairs. Then* $M = \bigoplus_{\alpha\in\mathcal{J}} N_\alpha$ *is endo-reduced. Further, the R-module* $\prod_{\alpha\in\mathcal{J}} N_\alpha$ *is an endo-reduced module.*

Proof. Since N_α and N_β for all $\alpha, \beta \in \mathcal{J}$ are non-isomorphic simple modules, this implies $\mathrm{Hom}_R(N_\alpha, N_\beta) = 0$. Furthermore, each N_α is an endo-reduced module because each $\mathrm{End}_R(N_\alpha), \alpha \in \mathcal{J}$, is a division ring. Hence $M = \bigoplus_{\alpha\in\mathcal{J}} N_\alpha$ is endo-reduced by Theorem 1. The last part holds true since $\mathrm{End}_R(\prod_{\alpha\in\mathcal{J}} N_\alpha) \cong \prod_{\alpha\in\mathcal{J}} \mathrm{End}_R(N_\alpha) \cong \mathrm{End}_R(\bigoplus_{\alpha\in\mathcal{J}} N_\alpha) \cong \mathrm{End}_R(M)$ is a reduced ring of endomorphisms of the module $\prod_{\alpha\in\mathcal{J}} N_\alpha$. □

Proposition 8. *Let* $\{(R_\alpha, M_\alpha) \mid \alpha \in \mathcal{J}\}$ *be a non-empty family of pairs with each* M_α *an* R_α*-module and let* $R = \prod_{\alpha\in\mathcal{J}} R_\alpha$. *Then* $\bigoplus_{\alpha\in\mathcal{J}} M_\alpha$ *is an endo-reduced* R*-module if and only if each* $M_\alpha, \alpha \in \mathcal{J}$ *is endo-reduced.*

Proof. In view of Proposition 1, Theorem 1 and the fact that $\mathrm{End}_{R_\alpha}(M_\alpha) = \mathrm{End}_R(M_\alpha)$ for each $\alpha \in \mathcal{J}$, it is enough to show that $\mathrm{Hom}_R(M_\alpha, M_\beta) = 0$ whenever $\alpha \neq \beta$. Let $\varphi \in \mathrm{Hom}_R(M_\alpha, M_\beta)$. Since $\mathrm{Ann}_R(M_\alpha) + \mathrm{Ann}_R(M_\beta) = R$, it follows that $\varphi M_\alpha = \varphi M_\alpha \mathrm{Ann}_R(M_\alpha) = 0$, i.e. $\varphi = 0$. Then $\mathrm{End}_R(\bigoplus_{\alpha\in\mathcal{J}} M_\alpha) \cong \prod_{\alpha\in\mathcal{J}} \mathrm{End}_{R_\alpha}(M_\alpha)$ is a reduced ring if and only if $\mathrm{End}_{R_\alpha}(M_\alpha)$ is a reduced ring for each $\alpha \in \mathcal{J}$, and the proof is completed. □

Since every reduced ring is right and left non-singular (see [13, Lemma 7.8]; [20, Lemma 5.1]), it is natural to expect that an endo-reduced module will also have some kind of non-singularity. Unfortunately this is dashed by Example 2.6. A right R-module M is said to be *non-singular* if every $m \in M, mI = 0$ for an essential right ideal I of R implies that $m = 0$. It is *non-M-singular* (also called *polyform*) if for all submodules $K \subseteq M$ and nonzero R-homomorphisms $\varphi : K \to M, \ker(\varphi)$ is not essential in M. Rizvi–Roman [19] introduced the notion of $\mathcal{K}$-non-singularity. An R-module M is said to be *$\mathcal{K}$-non-singular* if, for all $\varphi \in \mathrm{End}_R(M), \mathrm{Ann}^r_M(\varphi) = \ker(\varphi) \subseteq^{\mathrm{ess}} M$ implies that $\varphi = 0$. In general, non-singular $\Rightarrow$ non-M-singular (polyform) $\Rightarrow$ $\mathcal{K}$-non-singular, but the reverse implications fail [19].

Example 2.6. Since for all $\overline{x} \in \mathbb{Z}_p, \overline{x} \cdot p\mathbb{Z} = 0$, and $p\mathbb{Z} \subseteq^{\mathrm{ess}} \mathbb{Z}$, the $\mathbb{Z}$-module $\mathbb{Z}_p$ is not non-singular (but $\mathbb{Z}_p$ is endo-reduced). The $\mathbb{Z}$-module $M := \mathbb{Q} \oplus \mathbb{Z}_2$ is endo-reduced by Example 2.4 but non-singular. Also, if we take $\mathbb{Z} \subseteq_{\mathbb{Z}} \mathbb{Q}$ and define $\varphi : \mathbb{Z} \to M$ by $\varphi z = (0, \overline{z})$, then the kernel of φ is $2\mathbb{Z} \subseteq^{\mathrm{ess}} \mathbb{Z}$, and hence M cannot be non-M-singular. Lastly, since the endomorphism $\varphi_p : \mathbb{Z}_{p^\infty} \to \mathbb{Z}_{p^\infty}$ of the endo-reduced $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ obtained by $\varphi_p x = xp$ for every $x \in \mathbb{Z}_{p^\infty}$ has a nonzero essential kernel $\ker(\varphi_p), \mathbb{Z}_{p^\infty}$ is never $\mathcal{K}$-non-singular.

We call M_R an *endo-domain module* if $\mathrm{End}_R(M)$ is a domain. It is called uniform if any two nonzero submodules of M intersect non-trivially. A relationship between non-singular modules and endo-reduced modules is reflected in the well-known result [10, Theorem 1.7] of Johnson and Wong. They proved that the endomorphism ring of a module which is uniform and non-singular is a domain; and if the module is quasi-injective, then the ring is a division ring. Example 2.7 gives a uniform module with a weaker form of non-singularity whose endomorphism ring is a domain.

Example 2.7. Let p be a prime number and let $M_{\mathbb{Z}} := \mathbb{Z}_p$. Clearly M is uniform since it is simple; and $\mathrm{End}_{\mathbb{Z}}(M)$ is a domain (in fact a division ring). However, as in Example 2.6, M is not non-singular. We remark that M is a $\mathcal{K}$-non-singular module.

Theorem 2 subsumes the Johnson-Wong result in [10, Theorem 1.7] and extends it to a larger class of $\mathcal{K}$-non-singular modules.

Theorem 2. *Let R be a ring and M be an R-module. If M is both uniform and $\mathcal{K}$-non-singular, then M is an endo-reduced module. In particular, the following statements hold:*

(1) *Every nonzero endomorphism of M is a monomorphism;*
(2) *M_R is an endo-domain module.*

Furthermore, if M is quasi-injective, then $End_R(M)$ is a division ring.

Proof. For the first part of the theorem, it is enough to prove (1) or (2).

(1) Let φ be a nonzero endomorphism of M. If $\ker(\varphi) \neq 0$, then $\ker(\varphi) \subseteq^{\text{ess}} M$ because M is uniform. Since M is $\mathcal{K}$-non-singular, $\varphi = 0$ and we have a contradiction. So φ is a monomorphism.

(2) Suppose that $\psi\varphi = 0$. If $\psi = 0$, then we are done. Otherwise, since $\psi\varphi M = 0$, $\varphi M = 0$ by (1). Thus $\varphi = 0$.

Next, assume that M is quasi-injective. By (1), it suffices to show that $0 \neq \varphi \in \text{End}_R(M)$ is an epimorphism. Let φ be an injective endomorphism of M. Then $\varphi M \cong M$, and so φM is M-injective. Thus φM is a direct summand of M; i.e. there exists some submodule N of M such that $M = \varphi M \oplus N$. Hence $M \cong M \oplus N$. Since M is endo-reduced, it is Dedekind finite by Lemma 1; thus $N = 0$. Therefore, $\varphi M = M$ and so, φ is an epimorphism. □

Remark 4. Recall that a module M is said to be Hopfian if every epimorphism φ of M is an isomorphism. Let φ be any surjective endomorphism of a module which is both uniform and $\mathcal{K}$-non-singular. By Theorem 2, φ is an automorphism of M. Hence every uniform and $\mathcal{K}$-non-singular module is Hopfian. Further, every uniform non-singular ring is a domain. This follows immediately from Theorem 2 when $M = R_R$.

Corollary 3. *Let $M = \bigoplus_{\alpha \in \mathcal{J}} M_\alpha$ be a direct sum of non-isomorphic indecomposable $\mathcal{K}$-non-singular quasi-injective modules. Then M is an endo-reduced module and $End_R(M)$ is a strongly regular ring.*

Proof. Let $\varphi_\alpha \in \text{End}_R(M_\alpha), \alpha \in \mathcal{J}$, be any nonzero endomorphism. Since each M_α is quasi-injective and indecomposable, it is uniform by [13, Exercise 6.32, p. 244]. Applying Theorem 2 to each M_α implies that φ_α is a automorphism of M_α for each $\alpha \in \mathcal{J}$. Then $\text{End}_R(M_\alpha)$ is a division ring for every $\alpha \in \mathcal{J}$ and $\text{Hom}_R(M_\alpha, M_\beta) = 0$ as M_α and M_β for all $\alpha, \beta \in \mathcal{J}$ are non-isomorphic. Since $\text{End}_R(M) = \text{End}_R(\bigoplus_{\alpha \in \mathcal{J}} M_\alpha) \cong \prod_{\alpha \in \mathcal{J}} \text{End}_R(M_\alpha), \text{End}_R(M)$ is a strongly regular ring. □

Definition 2.2. Let M be an R-module and $S = \text{End}_R(M)$. We call ${}_SM$ a *P-flat module* if for any $\varphi \in S, \text{Ann}^r_M(\varphi) = \text{Ann}^r_S(\varphi)(M)$. It is a *P-injective module* if

$\varphi M = \mathrm{Ann}_M^r(\mathrm{Ann}_S^l(\varphi M))$ for every $\varphi \in S$ (see [18] for both terminologies). A module M_R is said to be *retractable* if for every $0 \neq N \subseteq M, \mathrm{Hom}_R(M, N) \neq 0$.

The notions: "retractable module", "P-injective module" and "P-flat module" are independent from the notion of "endo-reduced module". Further, the retractable modules are not necessarily P-flat modules.

Example 2.8. (a) The $\mathbb{Z}$-module $\mathbb{Q}$ is endo-reduced. However, since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$ for the submodule $\mathbb{Z}, \mathbb{Q}_{\mathbb{Z}}$ is not retractable. On the other hand, it is easy to check that the $\mathbb{Z}$-module $\mathbb{Z}_4$ is both retractable and P-injective but, as in Example 2.4, not endo-reduced.

(b) $\mathbb{Z}_{p^\infty}$ is an endo-reduced $\mathbb{Z}$-module by Example 2.1. However, since for every $\varphi \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}), 0 = \mathrm{Ann}^r_{\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})}(\varphi)(\mathbb{Z}_{p^\infty}) \subsetneqq \mathrm{Ann}^r_{\mathbb{Z}_{p^\infty}}(\varphi) \subseteq^{\mathrm{ess}} \mathbb{Z}_{p^\infty}$, the module $\mathbb{Z}_{p^\infty}$ is never P-flat. On the hand, $\mathbb{Z}^{(\mathbb{N})}$ is a P-flat $\mathbb{Z}$-module which is not endo-reduced as a $\mathbb{Z}$-module (Example 2.3).

(c) Since $\mathrm{Ann}^r_{\mathbb{Q}_{\mathbb{Z}}}(\varphi) = 0$ for each $\varphi \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Q}) = \mathbb{Q}$, $\mathbb{Q}_{\mathbb{Z}}$ is a P-flat module which is not retractable (by (a)). For the endo-reduced $\mathbb{Z}$-modules $\mathbb{Z}_p$ and $\mathbb{Z}_{p^\infty}$ (where p is a prime number), the direct sum $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$ is retractable as a $\mathbb{Z}$-module but not P-flat over $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_p \oplus \mathbb{Z}_{p^\infty})$, see [14, Example 3.14].

The discourse in Example 2.8 implies that Proposition 9 is appropriate. The proposition shows that endo-reduced modules with P-flat (respectively, P-injectivity) property can be described in terms of kernels (respectively, images) of their endomorphisms.

Proposition 9. *Let R be a ring, M_R be a nontrivial R-module and $S = End_R(M)$:*

(1) *If either $_SM$ is P-flat or M_R is retractable, then M_R is endo-reduced if and only if $Ann^r_M(\varphi) = Ann^r_M(\varphi^2)$ for every nonzero endomorphism $\varphi \in S$;*
(2) *If $_SM$ is P-injective, then M_R is endo-reduced if and only if $\varphi M = \varphi^2 M$ for every nonzero endomorphism $\varphi \in S$;*
(3) *$_SM$ is P-flat and M_R is endo-domain if and only if $Ann^r_M(\varphi) = 0$ for every nonzero endomorphism $\varphi \in S$;*
(4) *$_SM$ is P-injective and M_R is endo-domain if and only if $\varphi M = M$ for every nonzero endomorphism $\varphi \in S$.*

Proof. (1) Assume that $_SM$ is P-flat and M_R is endo-reduced. Using Proposition 4, $\mathrm{Ann}^r_M(\varphi) = \mathrm{Ann}^r_S(\varphi)(M) = \mathrm{Ann}^r_S(\varphi^2)(M) = \mathrm{Ann}^r_M(\varphi^2)$. Hence $\mathrm{Ann}^r_M(\varphi) = \mathrm{Ann}^r_M(\varphi^2)$.

Assume that M_R is both retractable and endo-reduced. Let $\varphi \in S$ and $x \in M$ such that $\varphi^2 x = 0$. Suppose that $\varphi x \neq 0$. Then $\varphi x \in \mathrm{Ann}^r_M(\varphi) \cap \varphi M \neq 0$. By the retractability of M, there exists $0 \neq \phi \in \mathrm{Hom}_R(M, \mathrm{Ann}^r_M(\varphi) \cap \varphi M)$ such that $\phi M \subseteq \mathrm{Ann}^r_M(\varphi)$ and $\phi M \subseteq \varphi M$. Since $\varphi\phi M = 0, \phi\varphi M = 0$ by Remark 1(c). This gives $\varphi M \subseteq \mathrm{Ann}^r_M(\phi)$, and we deduced that $\phi M \subseteq \varphi M \subseteq \mathrm{Ann}^r_M(\phi)$ so that $\phi^2 M = 0$. As M is endo-reduced, $\phi = 0$; which creates a contradiction to the choice of ϕ. Therefore, $\varphi x = 0$. This proves that $\mathrm{Ann}^r_M(\varphi^2) \subseteq \mathrm{Ann}^r_M(\varphi)$. Since $\mathrm{Ann}^r_M(\varphi) \subseteq \mathrm{Ann}^r_M(\varphi^2)$ always holds, $\mathrm{Ann}^r_M(\varphi^2) = \mathrm{Ann}^r_M(\varphi)$. Conversely, let $\varphi \in S$ such that $\varphi^2 M = 0$. Then $M = \mathrm{Ann}^r_M(\varphi^2) = \mathrm{Ann}^r_M(\varphi) \Rightarrow \varphi M = 0 \Rightarrow \varphi = 0$. This proves that M is endo-reduced.

(2) Assume that ${}_SM$ is P-injective and M_R is endo-reduced. Using Proposition 4, $\varphi M = \mathrm{Ann}^r_M(\mathrm{Ann}^l_S(\varphi M)) = \mathrm{Ann}^r_M(\mathrm{Ann}^l_S(\varphi^2 M)) = \varphi^2 M$ for every $\varphi \in S$. Hence $\varphi M = \varphi^2 M$. Conversely, $\varphi^2 M = 0$ implies that $\varphi M = 0$ because $\varphi^2 M = \varphi M$. This gives $\varphi = 0$ and so, M is endo-reduced.

(3) Suppose ${}_SM$ is P-flat and M_R is endo-domain. For each nonzero endomorphism $\varphi \in S, \mathrm{Ann}^r_M(\varphi) = \mathrm{Ann}^r_S(\psi)(M) = 0(M) = 0$. So $\mathrm{Ann}^r_M(\varphi) = 0$. Conversely, let $\psi, \varphi \in S$ such that $\psi\varphi = 0$. If $\psi \neq 0$, then $\varphi M \subseteq \mathrm{Ann}^r_M(\psi) = 0$. So $\varphi = 0$ and M is endo-domain. This also implies that $\mathrm{Ann}^r_S(\psi) = 0$. Thus ${}_SM$ is P-flat.

(4) Given that ${}_SM$ is P-injective and M_R is endo-domain. Then $\varphi M = \mathrm{Ann}^r_M(\mathrm{Ann}^l_S(\varphi M)) = \mathrm{Ann}^r_M(\mathrm{Ann}^l_S(M)) = \mathrm{Ann}^r_M(0_S) = M$. So $\varphi M = M$. Conversely, let $\psi, \varphi \in S$ such that $\psi\varphi = 0$. If $\varphi \neq 0$, then $\psi M = \psi\varphi M = 0$. So $\psi = 0$. Thus M is endo-domain. Further, $\varphi M = M$ implies that $\mathrm{Ann}^r_M(\mathrm{Ann}^l_S(\varphi M)) = \mathrm{Ann}^r_M(\mathrm{Ann}^l_S(M)) = \mathrm{Ann}^r_M(0_S) = M = \varphi M$. So $\mathrm{Ann}^r_M(\mathrm{Ann}^l_S(\varphi M)) = \varphi M$ and thus ${}_SM$ is P-injective. □

Call M_R a *Rickart module* (respectively, *dual Rickart module*) if $\ker(\varphi) \subseteq^{\oplus} M$ (respectively, $\varphi M \subseteq^{\oplus} M$) for every $\varphi \in S$ [14, 15].

Corollary 4. *A Rickart module or a free module* (*respectively, dual Rickart module*) *M_R is endo-reduced if and only if $Ann^r_M(\varphi) = Ann^r_M(\varphi^2)$* (*respectively, $\varphi M = \varphi^2 M$*) *for every $\varphi \in End_R(M)$.*

Proof. Since free modules and Rickart modules (respectively, dual Rickart modules) are P-flat (respectively, P-injective) over $\mathrm{End}_R(M)$ by [18], the proof of the corollary follows from Proposition 9. □

We show when an endo-reduced module which is P-flat (respectively, retractable, P-injective) is a reduced module.

Theorem 3. *Let M_R be a nontrivial R-module and $S = End_R(M)$. If ${}_SM$ is P-flat* (*respectively, M_R is retractable*), *then the following statements* (1) *and* (2) *are equivalent*:

(1) (a) *M_R is an endo-reduced module*;
 (b) *If $\varphi\phi m = 0$, then $\phi\varphi m = 0$ for every $m \in M$ and $\varphi, \phi \in S$*;
(2) *${}_SM$ is a reduced module.*

Proof. (1)⇒(2) Suppose that (1) holds. By Proposition 9, $\mathrm{Ann}^r_M(\varphi) = \mathrm{Ann}^r_M(\varphi^2)$ for every $\varphi \in S$. Then by Proposition 2, ${}_SM$ is a reduced module.

(2)⇒(1) (a) follows from Example 2.1 and (b) follows from Proposition 2. □

Theorem 4. *Let M_R be a nontrivial R-module and $S = End_R(M)$. The following statements are equivalent for a P-injective module ${}_SM$:*

(1) *M_R is an endo-reduced module and $Ann^r_M(\varphi) \cap \varphi M = 0$ for each $\varphi \in S$;*
(2) *${}_SM$ is a reduced module;*
(3) *$M_R = \varphi M \oplus \ker(\varphi)$ for each $\varphi \in S$.*

Proof. (1)⇒(3) Given (1). From Proposition 9 and the well-known (and easily proved) fact that $M = \varphi M + \ker(\varphi)$ if and only if $\varphi M = \varphi^2 M$, we have $M = \varphi M \oplus \ker(\varphi)$.

(3)⇒(2) In view of Proposition 2, we only prove that $\phi\varphi m = 0$ implies $\varphi\phi m = 0$ for every $\varphi, \phi \in S$ and $m \in M_R$. Indeed, $\phi\varphi m = 0$ implies $\varphi m \in \ker(\phi) \cap \phi M = 0$, so $\varphi m = 0$. But again, $m \in \ker(\varphi) \cap \varphi M = 0 \Rightarrow m = 0$. This implies that $\varphi\phi m = 0$. Hence ${}_SM$ is a reduced module.

(2)⇒(1) Follows from Proposition 2 and Example 2.1(d). □

Remark 5. The hypotheses in Theorems 3 and 4 are not superfluous.

(a) (M_R is endo-reduced + $\varphi\phi m = 0 \Rightarrow \phi\varphi m = 0$) $\not\Rightarrow$ ${}_SM$ is reduced. The $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$ is endo-reduced with $S = \mathrm{End}_{\mathbb{Z}}(M)$ commutative. Hence $\varphi\phi m = 0 \Rightarrow \phi\varphi m = 0\ \forall \varphi, \phi \in S, m \in M$; while ${}_SM$ is not reduced by Example 2.2.
(b) (M_R is endo-reduced + $\mathrm{Ann}^r_M(\varphi) \cap \varphi M = 0$) $\not\Rightarrow$ $M_R = \varphi M \oplus \ker(\varphi)$. Since the $\mathbb{Z}$-module $\mathbb{Z}$ is endo-reduced but not P-injective, $\mathbb{Z}_{\mathbb{Z}} \neq \varphi\mathbb{Z} \oplus \ker(\varphi)$. Note that for the endomorphism $\varphi : \mathbb{Z}_{\mathbb{Z}} \to \mathbb{Z}_{\mathbb{Z}}$ defined by $\varphi(n) = 2n$ for all $n \in \mathbb{Z}$, we have $\mathrm{Ann}^r_{\mathbb{Z}}(\varphi) \cap \varphi\mathbb{Z} = 0 \cap 2\mathbb{Z} = 0$ while $\mathbb{Z}_{\mathbb{Z}} \neq 2\mathbb{Z} = \varphi\mathbb{Z} + \mathrm{Ann}^r_{\mathbb{Z}}(\varphi)$.
(c) (${}_SM$ is P-injective + M_R is endo-reduced) $\not\Rightarrow$ ${}_SM$ is reduced. The $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$ (as in (a)) is endo-reduced and ${}_SM$ is P-injective, while ${}_SM$ is not reduced. Note that $\mathrm{Ann}^r_M(\varphi) \cap \varphi M \neq 0$ for all $0 \neq \varphi \in S$.
(d) A P-injective module ${}_SM$ is reduced if and only if $S = \mathrm{End}_R(M)$ is a strongly regular ring (Theorem 4 and [15, Proposition 2.22]).

An R-module M is *(quasi-)duo* if every (maximal) submodule of M is fully invariant, i.e. for any (maximal) submodule N of $M, \varphi N \subseteq N$ for every $\varphi \in \mathrm{End}_R(M)$. Let $\{M_\alpha \,|\, \alpha \in \mathcal{J}\}$ be a family of R-modules. A module M is said to be a *subdirect product* of $\{M_\alpha \,|\, \alpha \in \mathcal{J}\}$, if M is a submodule of $\prod_{\alpha \in \mathcal{J}} M_\alpha$ such that for every $\alpha \in \mathcal{J}$ the canonical projection π_α restricted to M is an epimorphism.

Proposition 10. *Let M be a quasi-duo R-module. Then $M/\mathcal{R}ad(M)$ is an endo-reduced module. Moreover, $End_R(M/\mathcal{R}ad(M))$ is isomorphic to a direct product of division rings. If in addition M is quasi-projective, then $End_R(M)/\mathcal{R}ad(End_R(M))$ is a reduced ring.*

Proof. Let $\{S_\alpha \,|\, \alpha \in \mathcal{J}\}$ ($\mathcal{J}$ is an arbitrary index set) be a family of simple R-modules, and consider a monomorphism

$$\vartheta : M/\mathcal{R}ad(M) \to \prod_{\alpha\in\mathcal{J}} S_\alpha.$$

For each $\alpha \in \mathcal{J}$, consider the restriction of the projection π_α to $\vartheta(M/\mathcal{R}ad(M))$

$$\pi_\alpha|_{\vartheta(\frac{M}{\mathcal{R}ad(M)})} : \vartheta\left(\frac{M}{\mathcal{R}ad(M)}\right) \to S_\alpha.$$

Since $(\frac{M}{\mathcal{R}ad(M)})/\ker(\pi_\alpha\vartheta) \cong \vartheta(\frac{M}{\mathcal{R}ad(M)})/\ker(\pi_\alpha|_{\vartheta(\frac{M}{\mathcal{R}ad(M)})}) \cong S_\alpha$ for each $\alpha \in \mathcal{J}$, it follows that $\ker(\pi_\alpha\vartheta)$ is a maximal submodule of $M/\mathcal{R}ad(M)$. As M is quasi-duo, $\ker(\pi_\alpha\vartheta)$ is fully invariant in $M/\mathcal{R}ad(M)$. Hence every R-endomorphism $\gamma : M/\mathcal{R}ad(M) \to M/\mathcal{R}ad(M)$ induces a well-defined R-module endomorphism $\gamma_\alpha : S_\alpha \to S_\alpha$ for some $\alpha \in \mathcal{J}$. This defines a ring homomorphism

$$\Phi : \mathrm{End}_R(M/\mathcal{R}ad(M)) \to \prod_{\alpha\in\mathcal{J}} \mathrm{End}_R(S_\alpha), \quad \gamma \mapsto (\gamma_\alpha).$$

Assume that $\gamma^2 = 0$. Then $0 = \Phi(0) = \Phi(\gamma^2) = \Phi(\gamma)^2 = (\gamma_\alpha)^2 = (\gamma_\alpha^2)$. So $\gamma_\alpha^2 = 0$ for each $\alpha \in \mathcal{J}$. Since, by Example 2.1, S_α is endo-reduced for each $\alpha \in \mathcal{J}$, we have $\gamma_\alpha = 0$ for all $\alpha \in \mathcal{J}$. It follows that $0 = (0) = (\gamma_\alpha) = \Phi(\gamma)$ so that

$$\gamma \in \ker(\Phi) = \mathrm{Hom}_R\left(M/\mathcal{R}ad(M), \bigcap_{\alpha\in\mathcal{J}} \ker(\pi_\alpha\vartheta)\right).$$

We proceed to show that $\gamma = 0$. Indeed, $\pi_\alpha\vartheta\gamma = 0$ for each $\alpha \in \mathcal{J}$. This implies that $\vartheta\gamma = 0$. Since ϑ is a monomorphism, $\gamma = 0$, and so $\mathrm{End}_R(M/\mathcal{R}ad(M))$ is a reduced ring. We conclude that $M/\mathcal{R}ad(M)$ is an endo-reduced module. The second statement follows from the proceeding discussion that $\ker(\Phi) = 0$. Since Φ is a monomorphism, $\mathrm{End}_R(M/\mathcal{R}ad(M))$ is a subring of a direct product of division rings. Lastly, if in addition M is a quasi-projective module, then by [21, Sec. 22.2],

$$\mathrm{End}_R(M)/\mathcal{R}ad(\mathrm{End}_R(M)) \cong \mathrm{End}_R(M/\mathcal{R}ad(M)).$$

Thus $\mathrm{End}_R(M)/\mathcal{R}ad(\mathrm{End}_R(M))$ is a reduced ring. □

Corollary 5. *Let M be a quasi-duo module. If $\mathcal{R}ad(M) = 0$, then M is an endo-reduced module.*

It is well known that a subdirect product of reduced rings is a reduced ring. Proposition 11 provides a mild analogous module theoretic result for quasi-duo modules — using a subdirect product of some endo-reduced modules. Recall that simple modules are endo-reduced modules.

Proposition 11. *Consider the following statements for a module M:*

(1) *M is both a quasi-duo R-module and a subdirect product of simple modules*;
(2) *M is an endo-reduced R-module with $\mathcal{R}ad(M) = 0$;*
(3) *M is an Abelian module with $\mathcal{R}ad(M) = 0$.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3)*. If* M *is* $M^{(\mathcal{J})}$*-projective for every index set* $\mathcal{J}$*, then* (3) $\Rightarrow$ (1) *even without the Abelian hypothesis.*

Proof. (1) $\Rightarrow$ (2) Let $\{S_\alpha \,|\, \alpha \in \mathcal{J}\}$ ($\mathcal{J}$ arbitrary) be an indexing set of simple R-modules and M be a subdirect product of simple modules. Then there exists a monomorphism $\vartheta : M \to \prod_{\alpha\in\mathcal{J}} S_\alpha$ such that for each $\alpha \in \mathcal{J}$, the restriction of the projection π_α to M, $\pi_\alpha|_M : M \to S_\alpha$, is an epimorphism. Since $\ker(\pi_\alpha\vartheta)$ is maximal in M for each $\alpha \in \mathcal{J}$ and $\bigcap_{\alpha\in\mathcal{J}} \ker(\pi_\alpha\vartheta) = \ker(\vartheta) = 0$, it follows that

$$\mathcal{R}ad(M) \subseteq \bigcap_{\alpha\in\mathcal{J}} \ker(\pi_\alpha\vartheta) = 0$$

and so, $\mathcal{R}ad(M) = 0$. As M is quasi-duo, applying Corollary 5, M is an endo-reduced module.

(2) $\Rightarrow$ (3) M is an Abelian module by Lemma 1.

Assume that M is $M^{(\mathcal{J})}$-projective for every index set $\mathcal{J}$.

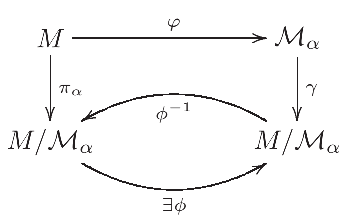
(3) $\Rightarrow$ (1) Since $\mathcal{R}ad(M) = 0, M$ contains at least one nonzero maximal submodule. Let $\{\mathcal{M}_\alpha \mid \alpha \in \mathcal{J}\}$ be an indexing of the maximal submodules of M. Since $M/\mathcal{M}_\alpha$ is simple for each $\alpha \in \mathcal{J}$

$$\bigcap_{\alpha\in\mathcal{J}} \{\ker(\gamma_\alpha) \,|\, \gamma_\alpha \in \mathrm{Hom}_R(M, M/\mathcal{M}_\alpha)\} = \mathrm{Rej}_M(\{M/\mathcal{M}_\alpha \,|\, \alpha \in \mathcal{J}\}) = \mathcal{R}ad(M) = 0.$$

Since $\{\gamma_\alpha \,|\, \alpha \in \mathcal{J}\}$ are epimorphisms, M must be a subdirect product of simple modules. To prove that M is quasi-duo, choose $\alpha \in \mathcal{J}$ such that for the maximal submodule $\mathcal{M}_\alpha$ we write

$$\mathcal{P} := \bigcap_{\gamma\in\mathrm{Hom}_R(M,M/\mathcal{M}_\alpha)} \ker(\gamma).$$

Then $\mathcal{P} = \mathrm{Rej}_M(M/\mathcal{M}_\alpha)$, the reject of $M/\mathcal{M}_\alpha$ in M. Let $\mathcal{A} = \mathrm{Hom}_R(M, \mathcal{M}_\alpha)$. Then $\mathcal{M}_\alpha = \sum\{\varphi M \,|\, \varphi \in \mathcal{A}\}$. We show that $\mathcal{A}\mathcal{M}_\alpha \subseteq \mathcal{P}$. Let $\varphi \in \mathcal{A}$ be any homomorphism and consider the composite homomorphism $\gamma\varphi : M \to M/\mathcal{M}_\alpha$. Since $\mathcal{M}_\alpha$ is a maximal submodule in $M, \gamma\varphi$ factors through the canonical projection $\pi_\alpha : M \to M/\mathcal{M}_\alpha$



This means that there exists an endomorphism $\phi : M/\mathcal{M}_\alpha \to M/\mathcal{M}_\alpha$ such that $\phi\pi_\alpha = \gamma\varphi$. Moreover, ϕ is an automorphism of $M/\mathcal{M}_\alpha$ because $M/\mathcal{M}_\alpha$ is simple. Therefore, it follows that $\ker(\gamma\varphi) = \ker(\phi\pi_\alpha) = \ker(\pi_\alpha) = \mathcal{M}_\alpha$. This implies that $(\gamma\varphi)M \subseteq \mathcal{M}_\alpha$ and so, $\varphi\mathcal{M}_\alpha \subseteq \varphi M \subseteq \ker(\gamma)$. Since this is true for every

$\gamma \in \text{Hom}_R(M, M/\mathcal{M}_\alpha), \varphi\mathcal{M}_\alpha \subseteq \mathcal{P}$. As this holds for every $\varphi \in \mathcal{A}$, we obtain $\mathcal{A}\mathcal{M}_\alpha \subseteq \mathcal{P}$.

Next, we show that $\mathcal{M}_\alpha \subseteq \mathcal{P}$. From $\mathcal{A}\mathcal{M}_\alpha \subseteq \mathcal{P}$, it is easy to see that $\text{Hom}_R(M, \mathcal{A}\mathcal{M}_\alpha)\frac{M}{\mathcal{M}_\alpha} = 0$. Since M is $M^{(\mathcal{J})}$-projective for every index set $\mathcal{J}$, in view of [21, Sec. 18.3]; [2, Proposition 5.6], $\mathcal{A}(\mathcal{A}\frac{M}{\mathcal{M}_\alpha}) = 0$. Therefore, $\mathcal{A}\frac{M}{\mathcal{M}_\alpha} = 0$ or $\mathcal{A}\frac{M}{\mathcal{M}_\alpha} = M/\mathcal{M}_\alpha$. If $\mathcal{A}\frac{M}{\mathcal{M}_\alpha} = 0$, then $\mathcal{M}_\alpha \subseteq \mathcal{P}$. If $\mathcal{A}\frac{M}{\mathcal{M}_\alpha} = M/\mathcal{M}_\alpha$, then $\mathcal{A}\frac{M}{\mathcal{M}_\alpha} = \mathcal{A}(\mathcal{A}\frac{M}{\mathcal{M}_\alpha}) = 0$, which also gives $\mathcal{M}_\alpha \subseteq \mathcal{P}$. But $\mathcal{M}_\alpha$ is maximal, so $\mathcal{M}_\alpha = \mathcal{P}$. Since $\mathcal{P}$ is fully invariant in M by [21, Sec. 14.5], M is quasi-duo, completing the proof. □

Since the subdirect product of simple modules $M = \mathbb{R} \oplus \mathbb{R}$ in Example 2.5(a) is not quasi-duo, M is not an endo-reduced module. Hence the hypothesis that M is a quasi-duo module in (1) ⇒ (2) of Proposition 11 is not superfluous.

3. Endo-Reduced Multiplication Modules

An R-module M is a *multiplication module* if for each R-submodule N of $M, N = MI$ for some ideal I of R. It is well known that an R-module M is a multiplication module if and only if $N = M(N : M)$ for all submodules N of M [7]. Examples of multiplication R-modules include the following: projective ideals of R, ideals of R generated by idempotents, projective duo R-modules and finitely generated R-modules M such that every localization of R with respect to a maximal ideal of R is cyclic. For finitely generated multiplication modules, we show that M is a reduced S-module ⇔ M is an endo-reduced R-module ⇔ M is a reduced R-module, where $S = \text{End}_R(M)$.

Lemma 3. *Let R be a commutative ring, M be a finitely generated R-module and $a \in R$. Then*

(1) $M(a) = M(a^2)$ *if and only if* $(a) + Ann_R(M) = (a^2) + Ann_R(M)$;
(2) $Ann_{\frac{R}{Ann_R(M)}}((a) + Ann_R(M)) = Ann_{\frac{R}{Ann_R(M)}}((a^2) + Ann_R(M))$ *if and only if* $Ann_R(M(a)) = Ann_R(M(a^2))$.

Proof. (1) Suppose that $M(a) = M(a^2)$. Then $M(a) = M(a)(a)$ where $M(a)$ is a finitely generated module. Using the Nakayama's lemma (see [1, Corollary 2.5]), we have $M(a)(1+(a)) = 0$ which implies that $(1 + ar) \in \text{Ann}_R(M(a))$ for every $r \in R$. So $R = (a) + \text{Ann}_R(M(a))$ and hence $(a) = (a^2)$. Therefore, $(a) + \text{Ann}_R(M) = (a^2) + \text{Ann}_R(M)$. Conversely, if $(a) + \text{Ann}_R(M) = (a^2) + \text{Ann}_R(M)$, then it is easy to see that $M(a) = M(a^2)$.

(2) Assume $\text{Ann}_{\frac{R}{\text{Ann}_R(M)}}((a) + \text{Ann}_R(M)) = \text{Ann}_{\frac{R}{\text{Ann}_R(M)}}((a^2) + \text{Ann}_R(M))$ and let $x \in \text{Ann}_R(M(a^2))$. Then $xa^2 \in \text{Ann}_R(M)$. It follows that $(x + \text{Ann}_R(M)) \in \text{Ann}_{\frac{R}{\text{Ann}_R(M)}}((a^2) + \text{Ann}_R(M)) = \text{Ann}_{\frac{R}{\text{Ann}_R(M)}}((a) + \text{Ann}_R(M))$. Therefore, $xa \in \text{Ann}_R(M)$ so that $x \in \text{Ann}_R(M(a))$. Hence $\text{Ann}_R(M(a^2)) \subseteq \text{Ann}_R(M(a))$. As

$\mathrm{Ann}_R(M(a)) \subseteq \mathrm{Ann}_R(M(a^2))$ always holds, we get $\mathrm{Ann}_R(M(a)) = \mathrm{Ann}_R(M(a^2))$. Conversely, assume $\mathrm{Ann}_R(M(a)) = \mathrm{Ann}_R(M(a^2))$ and let $(x + \mathrm{Ann}_R(M)) \in \mathrm{Ann}_{\frac{R}{\mathrm{Ann}_R(M)}}((a^2)+\mathrm{Ann}_R(M))$. Then $xa^2 \in \mathrm{Ann}_R(M)$ such that $x \in \mathrm{Ann}_R(Ma^2) = \mathrm{Ann}_R(Ma)$. Then $xa \in \mathrm{Ann}_R(M)$ so that $(x + \mathrm{Ann}_R(M)) \in \mathrm{Ann}_{\frac{R}{\mathrm{Ann}_R(M)}}((a) + \mathrm{Ann}_R(M))$. This gives $\mathrm{Ann}_{\frac{R}{\mathrm{Ann}_R(M)}}((a^2) + \mathrm{Ann}_R(M)) \subseteq \mathrm{Ann}_{\frac{R}{\mathrm{Ann}_R(M)}}((a) + \mathrm{Ann}_R(M))$. Since the reverse inclusion is easy to see, the proof of the lemma is completed. □

Lemma 4. *Let R be a commutative ring, and let M be a finitely generated R-module. Then the following statements are equivalent*:

(1) *$R/Ann_R(M)$ is a reduced ring*;
(2) *$Ann_R(Ma) = Ann_R(Ma^2)$ for each $a \in R$*;
(3) *$Ann_R(MI) = Ann_R(MI^2)$ for all ideals I of R.*

Proof. (1)⇔(2) Applying Proposition 3, $R/\mathrm{Ann}_R(M)$ is a reduced ring if and only if $\mathrm{Ann}_{\frac{R}{\mathrm{Ann}_R(M)}}((a)+\mathrm{Ann}_R(M)) = \mathrm{Ann}_{\frac{R}{\mathrm{Ann}_R(M)}}((a^2)+\mathrm{Ann}_R(M))$ for every $a + \mathrm{Ann}_R(M) \in R/\mathrm{Ann}_R(M)$. Using Lemma 3, this holds if and only if $\mathrm{Ann}_R(M(a)) = \mathrm{Ann}_R(M(a^2))$.

(2)⇒(3) Let I be an ideal of R. Then in general, $\mathrm{Ann}_R(MI) \subseteq \mathrm{Ann}_R(MI^2)$. Using the result in (2) and the fact that $\sum_{a\in I} Ma^2 \subseteq MI^2$, $\mathrm{Ann}_R(MI^2) \subseteq \mathrm{Ann}_R(\sum_{a\in I} Ma^2) = \bigcap_{a\in I} \mathrm{Ann}_R(Ma^2) = \bigcap_{a\in I} \mathrm{Ann}_R(Ma) = \mathrm{Ann}_R(\sum_{a\in I} Ma) = \mathrm{Ann}_R(MI)$. Hence $\mathrm{Ann}_R(MI) = \mathrm{Ann}_R(MI^2)$.

(3)⇒(2) Since for every $a \in R, (a)$ is an ideal of R, we have $\mathrm{Ann}_R(Ma) = \mathrm{Ann}_R(M(a)) = \mathrm{Ann}_R(M(a^2)) = \mathrm{Ann}_R(Ma^2)$. Hence $\mathrm{Ann}_R(Ma) = \mathrm{Ann}_R(Ma^2)$. □

For commutative rings, it can be easily verified that M_R is a reduced module if and only if for each $m \in M$ and $a \in R, ma^2 = 0$ implies $ma = 0$. Proposition 12 illustrates when the notions: "${}_SM$ is a reduced module", "M_R is an endo-reduced module" and "M_R is a reduced module" are indistinguishable (where $S = \mathrm{End}_R(M)$).

Proposition 12. *Let R be a commutative ring, and let M be a finitely generated multiplication R-module. Then the following statements are equivalent*:

(1) *M_R is an endo-reduced module*;
(2) *${}_SM$ is a reduced module where $S = End_R(M)$*;
(3) *M_R is a reduced module*;
(4) *$R/Ann_R(M)$ is a reduced ring.*

Proof. (1)⇒(4) Follows from Remark 1.

(4)⇒(3) Suppose $ma^2 = 0$ for some $a \in R$ and $m \in M$. Then we have $mRa^2 = 0$. Since M is a multiplication module, we have $mR = M(mR : M)$. This implies that

$0 = mRa^2 = M(mR : M)a^2$ and so $(mR : M) \subseteq \mathrm{Ann}_R(Ma^2)$. Since $R/\mathrm{Ann}_R(M)$ is a reduced ring, by Lemma 4, $(mR : M) \subseteq \mathrm{Ann}_R(Ma)$. It follows that $mRa = M(mR : M)a = 0$ and hence $ma = 0$. Thus M_R is a reduced R-module.

(3)⇒(2) In view of Proposition 2(3), it is enough to prove that $S = \mathrm{End}_R(M)$ is a commutative ring and $\mathrm{Ann}^r_M(\varphi) \cap \varphi M = 0$ for each $\varphi \in S$. Let $\varphi \in S$. As M is a multiplication module, there exists an ideal I of R such that $\varphi M = MI$. In particular, for every $m \in M$ there exists some $a \in I$ such that $\varphi m = ma$. Let $x \in \mathrm{Ann}^r_M(\varphi) \cap \varphi M$. Then $x = \varphi m = ma$. So $0 = \varphi x = \varphi^2 m = ma^2$. Since M_R is a reduced module, $0 = ma = x$ and hence $\mathrm{Ann}^r_M(\varphi) \cap \varphi M = 0$. To prove commutativity of S, choose $\phi \in S$ in a similar manner as φ. Then $\phi m = mb$ for some b in some ideal J of R. Now, commutativity of R implies $\varphi\phi m = \varphi mb = (ma)b = (mb)a = \phi ma = \phi\varphi m$. This proves that S is a commutative ring. Therefore, ${}_SM$ is a reduced module.

(2)⇒(1) This follows from Example 2.1. □

Next, we characterize the classes of reduced rings and von Neumann regular rings in terms of endo-reduced modules.

Theorem 5. *The following statements are equivalent for a commutative ring R*:

(1) *Every finitely generated faithful multiplication R-module is an endo-reduced module*;
(2) *Every faithful cyclic R-module is an endo-reduced module*;
(3) *R is a reduced ring.*

Proof. (1)⇒(2) Follows easily since cyclic modules are finitely generated modules.

(2)⇒(3) Let M be a cyclic R-module which is endo-reduced. Since M_R is both a multiplication module and finitely generated module, $R/\mathrm{Ann}_R(M)$ is a reduced ring by Proposition 12. Hence R is a reduced ring because M is faithful.

(3)⇒(1) Let M be an R-module over a reduced ring R. Since M is faithful, $R/\mathrm{Ann}_R(M) \cong R$ is a reduced ring. Since M is a finitely generated multiplication module, by Proposition 12, M is an endo-reduced module. □

Theorem 6. *The following statements are equivalent for a commutative ring R*:

(1) *Every finitely generated multiplication R-module is an endo-reduced module*;
(2) *Every cyclic R-module is an endo-reduced module*;
(3) *R is a von Neumann regular ring.*

Proof. (1)⇒(2) This is clear.

(2)⇒(3) Let $a \in R$. Then $M_R := R/Ra^2$ is an endo-reduced cyclic R-module. Let $\varphi \in \mathrm{End}_R(M)$ be any endomorphism of M and $\varphi(1 + Ra^2) = a + Ra^2$. Since $\varphi^2 M = 0$ (where $0 = Ra^2$), $\varphi M = 0$ by (2). So $a + Ra^2 = \varphi M = Ra^2$. Therefore, $a = ra^2 = ara$ for some $r \in R \Rightarrow a$ is a von Neumann regular element of R, and so R is a von Neumann regular ring.

(3)⇒(1) Let R be a von Neumann regular ring and M be an R-module. Using [11, Corollary 2.2], $M = Ma \oplus \mathrm{Ann}^l_M(a)$ for each $a \in R$. That is, $\mathrm{Ann}^l_M(a) \cap Ma = 0$. Using Proposition 1 and the fact that R is a commutative ring, M_R is a reduced module. Applying Proposition 12, M is an endo-reduced module. □

Remark 6. Let R be a commutative ring.

(a) The statement "every (finitely generated) R-module is an endo-reduced module" does not hold for the rings considered in Theorems 5 and 6. Example 2.4 gives a (finitely generated) module $\mathbb{Z}_4$ over a reduced ring $\mathbb{Z}$ which is not endo-reduced. Example 2.3 shows that there is a (2-generated) module M_R, over a commutative von Neumann regular ring R, which is not endo-reduced.

(b) In view of [12, Corollary 20], a duo (recall that multiplication modules are duo) module with a von Neumann regular endomorphism ring is an endo-reduced module with a strongly regular endomorphism ring. Let $\varphi, \psi \in \mathrm{End}_R(M)$ and $m \in M$. Since M is a multiplication module, $\varphi m = ma$ and $\psi m = mb$ for some a and b in R. Since R is commutative, $\varphi\psi m = \varphi mb = mba = mab = \psi ma = \psi\varphi m$. Thus $\mathrm{End}_R(M)$ is a commutative regular ring. Consequently, M is an endo-reduced module with $\mathrm{End}_R(M)$ strongly regular.

4. Endo-Reduced Modules Over a Commutative Dedekind Domain

In this section, we investigate the endo-reduced modules over a commutative Dedekind domain. We show that an arbitrary direct sum of cyclic modules M over a commutative Dedekind domain D is an endo-reduced module if and only if M is either a semisimple module with pair-wise non-isomorphic submodules or M is a torsion-free module which is isomorphic to D_D (Theorem 5). Hence a finitely generated Abelian group G is an endo-reduced $\mathbb{Z}$-module if and only if $G \cong \bigoplus_{p_\alpha \in \mathcal{P}} \mathbb{Z}/p_\alpha\mathbb{Z}$, where $\mathcal{P}$ is a collection of distinct prime numbers p_α, or $G \cong \mathbb{Z}_\mathbb{Z}$ (Corollary 8).

We begin with a result on torsion-free modules over reduced rings. Note that such modules may not be endo-reduced (see Lemma 1 and Theorem 1). Proposition 13 provides a special case of when such modules are endo-reduced.

Proposition 13. *Let R be a reduced ring and $M = \bigoplus_{\alpha \in \mathcal{J}} M_\alpha$ be a direct sum of cyclic R-modules M_α over an arbitrary index set $\mathcal{J}$. If M is a torsion-free module, then M is an endo-reduced module if and only if M is isomorphic to R_R.*

Proof. Assume that M is torsion-free. Then each $M_\alpha, \alpha \in \mathcal{J}$, is a torsion-free R-module. Also for each $\alpha \in \mathcal{J}, M_\alpha = mR \cong R/\mathrm{Ann}_R(m)$ for some $0 \neq m \in M_\alpha$. But each M_α being torsion-free implies that $\mathrm{Ann}_R(m) = 0$ and so, $M_\alpha \cong R_R$ for each $\alpha \in \mathcal{J}$. Since R is a reduced ring, $M = \bigoplus_{\alpha \in \mathcal{J}} M_\alpha \cong \bigoplus_{\alpha \in \mathcal{J}} R_\alpha$ with each $R_\alpha \cong R_R$ endo-reduced. By applying Lemma 1 and Theorem 1, M is isomorphic to R_R. The converse follows easily. □

Corollary 6. (1) *A cyclic torsion-free module M over a domain D is endo-reduced if and only if $M_D \cong D_D$;*

(2) *The cyclic torsion-free endo-reduced $\mathbb{Z}$-modules are precisely the modules $n\mathbb{Z}$ with $n \in \mathbb{Z}$.*

Proof. Follows from Proposition 13. □

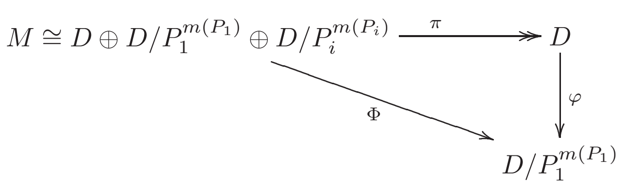
Lemma 5 ([1, Exercise 6, p. 99]). *Let M be a finitely generated torsion module over a commutative Dedekind domain D. Then M is uniquely representable as a finite direct sum of modules $D/P_\alpha^{n_\alpha}$, where $\alpha, n_\alpha \in \mathbb{N}$ and P_α are nonzero prime ideals of D.*

Theorem 7. *Let $M = \bigoplus_{\alpha\in\mathcal{J}} M_\alpha$ be a direct sum of cyclic D-modules M_α (where $\mathcal{J}$ is an arbitrary index set) over a commutative Dedekind domain D. Then the following statements are equivalent:*

(1) *M is an endo-reduced module;*
(2) *M is either a semisimple module with pair-wise non-isomorphic submodules or M is a torsion-free module which is isomorphic to D_D.*

Proof. (1)⇒(2) Assume $M = \bigoplus_{\alpha\in\mathcal{J}} M_\alpha$ is an endo-reduced D-module. Then we have a decomposition $M \cong \mathcal{T}(M) \oplus \mathcal{F}(M)$, where $\mathcal{T}(M)$ and $\mathcal{F}(M)$ are the torsion and the torsion-free submodules of M, respectively. Since each M_i is cyclic, $\mathcal{F}(M) \cong D$ by Proposition 13 and $\mathcal{T}(M) \cong \bigoplus_{P_i\in\mathcal{P}} D/P_i^{m(P_i)}$ by Lemma 5, where $\mathcal{P}$ is a (finite) collection of nonzero prime ideals P_i of D with $m(P_i) \in \mathbb{N}$ for each $P_i \in \mathcal{P}$.

We show that either $\mathcal{T}(M)$ or $\mathcal{F}(M)$ is trivial. Assume that $\mathcal{T}(M) \neq 0$ and $\mathcal{F}(M) \neq 0$. Since $\mathcal{T}(M) \neq 0, \mathcal{P} \neq \emptyset$. So there exist at least a nonzero prime ideal $P_1 \in \mathcal{P}$ with $m(P_1) \neq 0$ such that $M \cong D \oplus D/P_1^{m(P_1)} \oplus D/P_i^{m(P_i)}$. Consider the following diagram:



and define a canonical projection $\pi : M \to D$ by $\pi(r, (\bar{x})) = r$, where $r \in D$ and $(\bar{x}) \in D/P_1^{m(P_1)} \oplus D/P_i^{m(P_i)}$, and a D-morphism $\varphi : D \to D/P_1^{m(P_1)}$ by $\varphi r = \bar{r}$ where $\bar{r} := r + P_1^{m(P_1)}$. Then we can extend φ to an endomorphism Φ of M given by

$$\Phi = \varphi\pi.$$

Clearly $\Phi M \neq 0$ since $\varphi \neq 0$. However, for every $(r, (\bar{x})) \in M, \Phi^2(r, (\bar{x})) = \Phi\Phi(r, (\bar{x})) = \Phi\varphi\pi(r, (\bar{x})) = \Phi\varphi r = \Phi\bar{r} = \varphi\pi\bar{r} = \varphi 0 = 0$. Hence $\Phi^2 M = 0$, a contradiction. We conclude that either $\mathcal{T}(M) = 0$ or $\mathcal{F}(M) = 0$. Now consider

$\mathcal{F}(M) = 0$ such that $M = \mathcal{T}(M) \cong \bigoplus_{P_i \in \mathcal{P}} D/P_i^{m(P_i)}$ by Lemma 5. Then each $D/P_i^{m(P_i)}$ is an endo-reduced D-module by Proposition 5. Without loss of generality, we can assume that there exists a nonzero prime ideal P_i such that $m(P_i) > 1$. For such a module $D/P_i^{m(P_i)}$, define an endomorphism

$$\varphi_a : D/P_i^{m(P_i)} \to D/P_i^{m(P_i)} \quad \text{by } \varphi_a \bar{r} = \bar{r}a,$$

where $a \in P_i$ but $a \notin P_i^2$. Since $m(P_i) > 1$ and $\varphi_a \bar{1} = \bar{1} \cdot a = \bar{a} \neq 0$ modulo $P_i^{m(P_i)}$, such a φ_a is nonzero. Moreover, $D/P_i^{m(P_i)}$ is indecomposable with finite length (it is a fitting D-module) and $\mathrm{Im}(\varphi_a)$ properly imbeds in $D/P_i^{m(P_i)}$ (note that $0 \neq \mathrm{Im}(\varphi_a) \subset^{\text{ess}} D/P_i^{m(P_i)}$). So φ_a cannot be a D-epimorphism of $D/P_i^{m(P_i)}$. Using Lemma 2, the summand $D/P_i^{m(P_i)}$ fails to be an endo-reduced module; which creates a contradiction to Proposition 5. Hence $M \cong \bigoplus_{i \in \mathcal{J}, P_i \in \mathcal{P}} D/P_i$ with $\mathcal{P}$ a collection of prime ideals P_i of D. Moreover, due to Lemma 1 and Theorem 1, the prime ideals are not in multiple instances. So the D/P_i's are non-isomorphic. Since every nonzero prime ideal of a commutative Dedekind domain is maximal [1, p. 93], each D/P_i is a simple D-module; and thus M is a semisimple D-module with pair-wise non-isomorphic submodules. On the other hand, if we assume that $\mathcal{T}(M) = 0$, then $M = \mathcal{F}(M) \cong D_D$ by Proposition 13, which is a torsion-free module.

(2)⇒(1) Assume $M = \bigoplus_{\alpha \in \mathcal{J}} M_\alpha$ is a semisimple module whose submodules are non-isomorphic in pairs. Then M is endo-reduced by Corollary 2. Also if M is torsion-free and is isomorphic to D_D, then M is obviously an endo-reduced D-module. □

Corollary 7. *Let $M = \bigoplus_{\alpha \in \mathcal{J}} M_\alpha$ be a direct sum of finitely generated D-modules M_α (where $\mathcal{J}$ is an arbitrary index set) over a principal ideal domain D. Then the following conditions are equivalent:*

(1) *M is an endo-reduced module;*
(2) *M is either a semisimple module with pair-wise non-isomorphic submodules or M is isomorphic to D_D.*

Proof. Using the classical theorem for the decomposition of finitely generated modules over principal ideal domains, each M_α is a (finite) direct sum of cyclic submodules. Applying Theorem 7 completes the proof of the corollary. □

We characterize the endo-reduced Abelian groups (as $\mathbb{Z}$-modules) in the class of finitely generated Abelian groups and show that the endo-reduced finitely generated Abelian groups are exactly the $\mathbb{Z}$-modules: $n\mathbb{Z}$ with $n \in \mathbb{Z}$ or $\bigoplus_{p_\alpha \in \mathcal{P}} \mathbb{Z}/p_\alpha\mathbb{Z}$ where $\mathcal{P}$ is a collection of distinct prime numbers of $\mathbb{Z}$.

Corollary 8. *A finitely generated Abelian group G is endo-reduced as a $\mathbb{Z}$-module if and only if any one of the following two conditions holds*:

(1) $G \cong \bigoplus_{p_\alpha \in \mathcal{P}} \mathbb{Z}/p_\alpha\mathbb{Z}$, *where $\mathcal{P}$ is a collection of distinct prime numbers p_α of $\mathbb{Z}$ and $\alpha \in \mathbb{Z}^+$*;
(2) $G \cong \mathbb{Z}_\mathbb{Z}$.

Proof. Immediate from Corollary 7. □

Remark 7. (a) The $\mathbb{Z}$-module $\mathbb{Q}$ is a non-finitely generated (Abelian group) module that is endo-reduced but fails on both (1) and (2) of Corollary 8.
(b) The endo-reduced finitely generated torsion-free Abelian groups are precisely the modules $n\mathbb{Z}$ with $n \in \mathbb{Z}$. Hence using Corollaries 6 and 8, a finitely generated torsion-free Abelian group is endo-reduced as a $\mathbb{Z}$-module if and only if it is cyclic.

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