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OPTIMAL ACTUATOR PLACEMENT FOR CONTROL OF VIBRATIONS INDUCED BY PEDESTRIAN-BRIDGE INTERACTIONS

MARTIN DEOSBORNS AROP, HENRY KASUMBA, JUMA KASOZI, AND FREDRIK BERNTSSON

ABSTRACT. In this paper, an optimal actuator placement problem with a linear wave equation as the constraint is considered. In particular, this work presents the frameworks for finding the best location of actuators depending upon the given initial conditions, and where the dependence on the initial conditions is averaged out. The problem is motivated by the need to control vibrations induced by pedestrian-bridge interactions. An approach based on shape optimization techniques is used to solve the problem. Specifically, the shape sensitivities involving a cost functional are determined using the averaged adjoint approach. A numerical algorithm based on these sensitivities is used as a solution strategy. Numerical results are consistent with the theoretical results, in the two examples considered.

1. Introduction

An actuator is a device that introduces or prevents motion in a control system [10]. In this work, an actuator is defined as a device that prevents motion in a control system.

Optimal actuator placement problems involve the question of finding the optimal location of the subdomain [23]. They arise naturally in many practical applications, for example, in seismic inversion [20], placement of loudspeakers for ideal acoustics [11], and medical applications [2].

There are extensive works on the optimal actuator placement problems governed by linear ordinary differential equations in the literature, see [9, 22] and the references therein. From among the earlier publications in this direction, we quote the work in [9], where the optimal placement of actuators and sensors for gyroelastic bodies is studied based on controllability and observability criteria. Another important study is by Van de Wal and de Jager [22], where a linear system is solved using controllability and observability Gramians.

The optimal placement of actuators in dynamical systems governed by heat, advection, and wave equations has also received a growing amount of attention. In [21], an actuator and sensor placement problem is considered using an advection equation with an application in building systems. The authors proposed a Gramian criterion, where the degree of controllability and observability is maximized with respect to the least controllable and observable states.

An optimal actuator design and placement problem for a linear heat equation is investigated in [10] using a shape and topology optimization approach. The authors parametrized the actuators by considering controls over some subsets of the domains using indicator functions.

In [7] and [8], optimal stabilizations of the one-dimensional wave equation are investigated using a genetic algorithm and frequential analysis approach, respectively. Furthermore, the optimal location of controllers for the one-dimensional wave equation is studied in [16] as an exact controllability problem

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using the frequential analysis approach. In addition, the optimal location of the support of the control for the one-dimensional wave equation as an exact controllability problem is studied in [14].

Inspired by the work in [14], we study an optimal actuator placement problem for linear wave dynamics by using shape optimization techniques. In particular, we extend the techniques presented in [10] to a dynamic system governed by the linear wave equation. Numerical realization of the problem is achieved by using a finite difference method, see e.g., [13].

In this paper, we determine the optimal actuator placement for the stabilization of pedestrian-bridge vibrations. More precisely, we use a shape optimization approach to find the optimal actuator location so that the vibrations induced by pedestrian-bridge interactions are controlled.

The remainder of this paper is organized as follows. In Section 2, we fix the notations utilized in the sequel and formulate the state and optimization problems. Section 3 is devoted to proving well-posedness and deriving the optimality system for our optimization problems. In Section 4, we derive the shape derivatives of the optimization problems. Numerical tests that illustrate the theoretical results are given in Section 5. The paper ends with concluding remarks and future work.

2. Formulation of the Problem

2.1. **Notations.** Let $\mathcal{G}$ be either the domain Ω or its boundary $\partial\Omega$. Then, we define $L^2(\mathcal{G})$ as a linear space of all measurable functions $y : \mathcal{G} \to \mathbb{R}$ such that

$$\|y\|_{L^2(\mathcal{G})} := \left(\int_{\mathcal{G}} |y|^2 \, dx \right)^{\frac{1}{2}} < \infty.$$

The standard Sobolev space of order $m \in \mathbb{R}^+ \cup \{0\}$, denoted by $H^m(\mathcal{G})$, is defined as

$$H^m(\mathcal{G}) := \{y \in L^2(\mathcal{G}) | D^\gamma y \in L^2(\mathcal{G}), \text{ for all } 0 \le |\gamma| \le m\},$$

where D^γ is the weak partial derivative and γ is a multi-index. The norm $\|\cdot\|_{H^m(\mathcal{G})}$ associated with $H^m(\mathcal{G})$ is given by

$$\|y\|_{H^m(\mathcal{G})} := \sqrt{\sum_{|\gamma| \le m} \int_{\mathcal{G}} |D^\gamma y|^2 \, dx}.$$

For a functional space X, we denote by $L^p(0, T; X)$ $(1 \le p < \infty)$ the space of measurable functions $y : [0, T] \to X$ such that

$$\|y\|_{L^p(0,T;X)} := \left(\int_0^T \|y(\cdot, t)\|_X^p \, dt \right)^{\frac{1}{p}} < \infty,$$

where T is the final time. The space of essentially bounded functions from $[0, T]$ into X is denoted by $L^\infty(0, T; X)$ and is equipped with the norm $\operatorname{ess\,sup}_{t \in [0,T]} \|y(\cdot, t)\|_X$, where ess sup denotes the essential supremum. The duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ will be denoted by $\langle \cdot, \cdot \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$ while the inner product in $\mathbb{R}^2$ will be denoted by $(\cdot, \cdot)$. We denote the control space by $U := L^2(0, T; L^2(\Omega))$ and the collection of measurable subdomains of Ω by $E(\Omega)$. We shall use $L^2(L^2(\Omega)), L^2(H_0^1(\Omega))$ and $L^\infty(H_0^1(\Omega))$ as the short forms for $L^2(0, T; L^2(\Omega)), L^2(0, T; H_0^1(\Omega))$ and $L^\infty(0, T; H_0^1(\Omega))$, respectively.

2.2. **Setup of the Problem.** In this work, we consider the problem of controlling vibrations induced by pedestrian-bridge interactions, see Figure 1.

The vibrations $y(x, t)$ at position x and time t are governed by the wave equation:

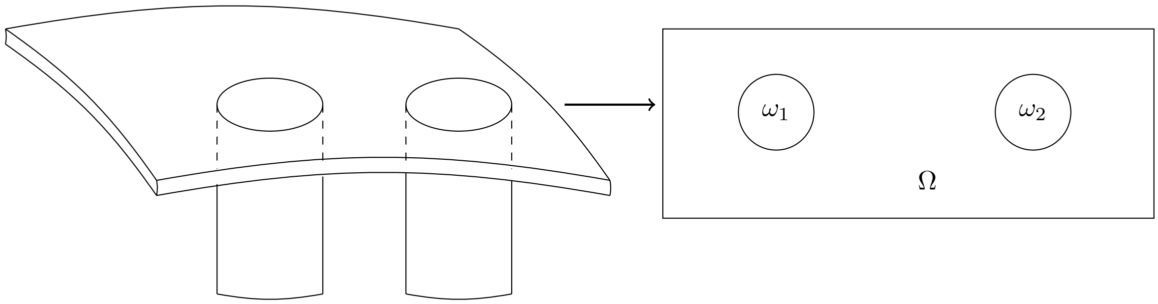


FIGURE 1. Control of vibrations on the domain Ω using the supports at $\omega := \omega_1 \cup \omega_2$.

$$
\begin{aligned}
&\frac{\partial^2 y}{\partial t^2} - \triangle y = \chi_\omega u, && (x,t) \in \Omega \times (0,T], \\
&y = 0, && (x,t) \in \partial\Omega \times (0,T], \\
&y(x,0) = f(x),\ \frac{\partial y}{\partial t}(x,0) = g(x),\ x \in \Omega,
\end{aligned}
\tag{2.1}
$$

where $u = u(x,t)$ denotes the control variable, χ_ω the characteristic function for the domain $\omega \subset \Omega$, and $x \in \mathbb{R}^2$. The domain ω represents the location of the actuators. It is not known where these supports should be placed in order to control the vibrations on the bridge. The goal is to determine the optimal location of these supports. The vibrations may depend on the initial conditions f and g, control variable u, and subdomain ω. This leads to the cost functional $J : E(\Omega) \times U_{ad} \times H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ defined by

$$
J(\omega, u, f, g) := \int_0^T \frac{1}{2}\|y^{u,f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\frac{dy^{u,f,g,\omega}}{dt}(\cdot,t)\right\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\chi_\omega u(\cdot,t)\|^2_{L^2(\Omega)} dt, \tag{2.2}
$$

where $\alpha > 0$ is a given parameter and U_{ad} is the admissible set of controls consisting of a closed and convex subset of U. The first and second terms in (2.2) suggest that we minimize the vibrations and speed, respectively while the third term is the control cost.

Remark 2.1. The notation $\chi_\omega u(x,t)$ is used to stress the fact that $u(x,t)$ is zero outside of ω.

Let ω, f and g be fixed. Then by taking the infimum of the cost J over all controls $u \in U_{ad}$, we obtain the functional $J_1 : E(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ defined by

$$
J_1(\omega, f, g) := \inf_{u \in U_{ad}} J(\omega, u, f, g). \tag{2.3}
$$

Note that the shape functional J_1 depends on the initial conditions f and g. To overcome such a dependence, we introduce a functional $J_2 : E(\Omega) \to \mathbb{R}$ defined by

$$
J_2(\omega) := \sup_{f \in K_1, g \in K_2} J_1(\omega, f, g), \tag{2.4}
$$

where K_1 and K_2 denote weakly compact subsets of $H_0^1(\Omega)$ and $L^2(\Omega)$ defined by

$$
K_1 := \{f : \|f\|_{H_0^1(\Omega))} \le 1\} \quad \text{and} \quad K_2 := \{g : \|g\|_{L^2(\Omega))} \le 1\},
$$

respectively. These conditions are used to average out the dependence of J_1 on the initial conditions, and overcome overflow for large values of f and g.

After introducing the two functionals in (2.3) and (2.4), we now study the problems of finding a minimum cost functional for a fixed $\omega \subset \Omega$ and a Lipschitz vector field $\mathbf{X}$.

Definition 2.1. The optimal actuator placement problems related to J_1 and J_2 are defined by the minimization problems:

$$\inf_{\mathbf{X}\in\mathbb{R}^2} J_1((\mathrm{id}+\mathbf{X})(\omega), f, g) \tag{2.5}$$

and

$$\inf_{\mathbf{X}\in\mathbb{R}^2} J_2((\mathrm{id}+\mathbf{X})(\omega)), \tag{2.6}$$

where $f \in K_1$, $g \in K_2$ and $(\mathrm{id}+\mathbf{X})(\omega) := \{x+\mathbf{X} : x \in \omega\}$, respectively.

3. Well-Posedness of the Functionals

To simplify the analysis, we reformulate the wave equation as a system. Note that by setting

$$\frac{\partial y^{u,f,g,\omega}}{\partial t} = v^{u,f,g,\omega},$$

we can rewrite (2.1) as the following first-order system:

$$\begin{cases} \frac{\partial y^{u,f,g,\omega}}{\partial t} - v^{u,f,g,\omega} = 0, & (x,t) \in \Omega \times (0,T], \\ \frac{\partial v^{u,f,g,\omega}}{\partial t} - \triangle y^{u,f,g,\omega} - \chi_\omega u = 0, & (x,t) \in \Omega \times (0,T], \\ y^{u,f,g,\omega}(x,0) = f(x),\ v^{u,f,g,\omega}(x,0) = g(x),\ x \in \Omega, & \\ y^{u,f,g,\omega} = 0, & (x,t) \in \partial\Omega \times (0,T]. \end{cases} \tag{3.1}$$

This reformulation is useful in the derivation of the optimality system and the discretization of the optimization problems.

The well-posedness of (3.1) and hence, (2.1) is guaranteed by the following Lemma:

Lemma 3.1. *Let $f \in H_0^1(\Omega), g \in L^2(\Omega)$ and $\chi_\omega u \in L^2(L^2(\Omega))$. Then the problem*

$$\begin{cases} \left\langle \frac{\partial v^{u,f,g,\omega}}{\partial t}, \phi \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_\Omega \nabla y^{u,f,g,\omega} \cdot \nabla \phi \, dx = \int_\Omega \chi_\omega u \phi \, dx, \\ \left(\frac{\partial y^{u,f,g,\omega}}{\partial t}, \psi \right) = \left(v^{u,f,g,\omega}, \psi \right), \end{cases} \tag{3.2}$$

for all $\phi \in L^2(H_0^1(\Omega))$ and $\psi \in L^2(L^2(\Omega))$ for a.e. $t \in (0,T]$ with $y^{u,f,g,\omega}(x,0) = f(x)$, $v^{u,f,g,\omega}(x,0) = g(x)$, has a unique weak solution $y^{u,f,g,\omega} \in L^2(H_0^1(\Omega))$ and $v^{u,f,g,\omega} \in L^2(L^2(\Omega))$ with

$$\frac{\partial v^{u,f,g,\omega}}{\partial t} \in L^2(H^{-1}(\Omega)).$$

Moreover, $y^{u,f,g,\omega} \in L^\infty(H^2 \cap H_0^1(\Omega))$ and $v^{u,f,g,\omega} \in L^\infty(H_0^1 \cap L^2(\Omega))$, and there exists a constant $c > 0$ that depends on Ω and T such that

$$\|y^{u,f,g,\omega}\|_{L^\infty(H_0^1(\Omega))} + \|v^{u,f,g,\omega}\|_{L^\infty(L^2(\Omega))} \le c\Big(\|\chi_\omega u\|_{L^2(L^2(\Omega))} + \|f\|_{H_0^1(\Omega)} + \|g\|_{L^2(\Omega)} \Big). \tag{3.3}$$

Proof. It is well known that problem (3.2) has a unique and stable weak solution $y^{u,f,g,\omega} \in L^\infty(H_0^1(\Omega)) \cap L^2(H_0^1(\Omega))$ and $v^{u,f,g,\omega} \in L^\infty(L^2(\Omega)) \cap L^2(L^2(\Omega))$, see e.g., [6, Chap. 7] . ☐

Now, we establish the convergence of the sequence of solutions to (3.1).

Lemma 3.2. *Suppose that $\{f_n\}$ is a sequence in K_1 that converges weakly in $H_0^1(\Omega)$ to $f \in K_1$, $\{g_n\}$ is a sequence in K_2 that converges weakly in $L^2(\Omega)$ to $g \in K_2$ and $\{u_n\}$ is a sequence in U_{ad} that converges weakly to a function $u \in U_{ad}$. Then:*

$$y^{u_n,f_n,g_n,\omega} \to y^{u,f,g,\omega} \text{ in } \; L^2(H_0^1(\Omega)) \text{ as } n \to \infty,$$
$$v^{u_n,f_n,g_n,\omega} \to v^{u,f,g,\omega} \text{ in } \; L^2(L^2(\Omega)) \text{ as } n \to \infty.$$

Proof. Note that inequality (3.3) implies that the sequences $\{y^{u_n,f_n,g_n,\omega}\}$ and $\{v^{u_n,f_n,g_n,\omega}\}$ are bounded in $L^2(H^2(\Omega) \cap H_0^1(\Omega))$ and $L^2(H_0^1(\Omega) \cap L^2(\Omega))$, respectively. By Rellich-Kondrachov theorem (see e.g.,[1]), we can extract the subsequences again denoted by $\{y^{u_n,f_n,g_n,\omega}\}$ and $\{v^{u_n,f_n,g_n,\omega}\}$ such that $\{y^{u_n,f_n,g_n,\omega}\}$ converges weakly to $y^{u,f,g,\omega}$ in $L^2(H^2(\Omega) \cap H_0^1(\Omega))$ and strongly to $y^{u,f,g,\omega}$ in $L^2(H_0^1(\Omega))$, and $\{v^{u_n,f_n,g_n,\omega}\}$ converges weakly to $v^{u,f,g,\omega}$ in $L^2(H_0^1(\Omega))$ and strongly to $v^{u,f,g,\omega}$ in $L^2(L^2(\Omega))$. Thus, replacing (u, f, g, ω) by (u_n, f_n, g_n, ω) in problem (3.2), we may pass to the limits and obtain by the uniqueness that $y = y^{u,f,g,\omega}$ and $v = v^{u,f,g,\omega}$. □

In the following lemma, we check that the optimization problem (2.3) is well-posed.

Lemma 3.3. *Problem (2.3) admits a unique optimal solution $\overline{u}$.*

Proof. We refer to [19, Chap. 1]. □

The notation $u^{f,g,\omega}$ will be used to indicate that u depends on f, g, ω.

Lemma 3.4. *Suppose that $\{f_n\}$ is a sequence in $H_0^1(\Omega)$ that converges weakly to f in $H_0^1(\Omega)$ and $\{g_n\}$ is a sequence in $L^2(\Omega)$ that converges weakly to g in $L^2(\Omega)$. Then we have*

$$\overline{u}^{f_n,g_n,\omega} \to \overline{u}^{f,g,\omega} \text{ in } U_{ad} \text{ as } n \to \infty,$$

where $\overline{u}^{f,g,\omega}$ solves (2.3).

Proof. Since $\overline{u}^{f_n,g_n,\omega}$ minimizes J with (ω, f, g) replaced by (ω, f_n, g_n), for all $u \in U_{ad}$ and $n \geq 0$, it follows from (3.3) that we must have

$$\begin{aligned}
&\frac{1}{2}\int_0^T \|y^{\overline{u}^{f_n,g_n,\omega},f_n,g_n,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \|v^{\overline{u}^{f_n,g_n,\omega},f_n,g_n,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \alpha\|\chi_\omega \overline{u}^{f_n,g_n,\omega}(\cdot,t)\|^2_{L^2(\Omega)} dt \\
&\leq \frac{1}{2}\int_0^T \|y^{u,f_n,g_n,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \|v^{u,f_n,g_n,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \alpha\|\chi_\omega u(\cdot,t)\|^2_{L^2(\Omega)} dt, \qquad (3.4)\\
&\leq c(\|\chi_\omega u\|^2_{L^2(L^2(\Omega))} + \|f_n\|^2_{H_0^1(\Omega)} + \|g_n\|^2_{L^2(\Omega)}).
\end{aligned}$$

This implies that $\{\overline{u}_n\} := \{\overline{u}^{f_n,g_n,\omega}\}$ is bounded in U_{ad}. By Rellich-Kondrachov theorem, we can extract a subsequence $\{\overline{u}_{n_k}\}$ such that $\overline{u}_{n_k} \rightharpoonup \overline{u}$ in U_{ad} as $k \to \infty$. Since $\overline{u}$ is a unique solution of $J(\omega, \cdot, f, g)$, the whole sequence $\{\overline{u}_n\}$ converges weakly to $\overline{u}$ in U_{ad} as $n \to \infty$. Thus, using Lemma 3.2 and by weak lower semicontinuity of norms, we may pass to the limit infimum in (3.4) to obtain

$$\begin{aligned}
&\int_0^T \|y^{\overline{u},f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \|v^{\overline{u},f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \alpha\|\chi_\omega \overline{u}(\cdot,t)\|^2_{L^2(\Omega)} dt \\
&\leq \int_0^T \|y^{u,f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \|v^{u,f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \alpha\|\chi_\omega u(\cdot,t)\|^2_{L^2(\Omega)} dt, \qquad (3.5)
\end{aligned}$$

for all $u \in U_{ad}$. So, we must have $\overline{u} = \overline{u}^{f,g,\omega}$ and since $\overline{u}^{f,g,\omega}$ is the minimizer of $J(\omega, \cdot, f, g)$ (see e.g., Lemma 3.3), the whole sequence $\{\overline{u}_n\}$ converges weakly to $\overline{u}^{f,g,\omega}$. Therefore, $\overline{u}_n \rightharpoonup \overline{u}^{f,g,\omega}$ in U_{ad}. As a consequence of weak lower semicontinuity, we must have
$\|\overline{u}^{f,g,\omega}\|_{L^2(L^2(\Omega))} \leq \lim_{k\to\infty} \inf \|\overline{u}_{n_k}\|_{L^2(L^2(\Omega))} = \|\overline{u}^{f,g,\omega}\|_{L^2(L^2(\Omega))}$.

Thus, it follows from (3.5) that the norm $\|\overline{u}^{f_n,g_n,\omega}\|_{L^2(L^2(\Omega))}$ converges to $\|\overline{u}^{f,g,\omega}\|_{L^2(L^2(\Omega))}$. The weak convergence and norm convergence of $(\overline{u}_n)$ imply that $\overline{u}^{f_n,g_n,\omega} \to \overline{u}^{f,g,\omega}$ in U_{ad} as $n \to \infty$. □

The following result will be used to characterize the optimal solution $\overline{u}$.

Theorem 3.5. *Suppose that $U_{ad} = U$. Then we have the following optimality system:*

$$\begin{array}{ll} \frac{\partial y^{\overline{u},f,g,\omega}}{\partial t} - v^{\overline{u},f,g,\omega} = 0, & (x,t) \in \Omega \times (0,T], \\ \frac{\partial v^{\overline{u},f,g,\omega}}{\partial t} - \triangle y^{\overline{u},f,g,\omega} - \chi_\omega \overline{u} = 0, & (x,t) \in \Omega \times (0,T], \\ y^{\overline{u},f,g,\omega}(x,0) = f,\ v^{\overline{u},f,g,\omega}(x,0) = g, & x \in \Omega, \\ y^{\overline{u},f,g,\omega} = 0, & (x,t) \in \partial\Omega \times (0,T], \end{array} \tag{3.6}$$

$$\begin{array}{ll} \frac{\partial p^{\overline{u},f,g,\omega}}{\partial t} - w^{\overline{u},f,g,\omega} = -v^{\overline{u},f,g,\omega}, & (x,t) \in \Omega \times (0,T], \\ \frac{\partial w^{\overline{u},f,g,\omega}}{\partial t} - \triangle p^{\overline{u},f,g,\omega} = -y^{\overline{u},f,g,\omega}, & (x,t) \in \Omega \times (0,T], \\ p^{\overline{u},f,g,\omega}(x,T) = 0,\ w^{\overline{u},f,g,\omega}(x,T) = 0, & x \in \Omega, \\ p^{\overline{u},f,g,\omega} = 0, & (x,t) \in \partial\Omega \times (0,T] \end{array} \tag{3.7}$$

and

$$\alpha\chi_\omega \overline{u} - \chi_\omega p^{\overline{u},f,g,\omega} = 0, \qquad (x,t) \in \Omega \times (0,T], \tag{3.8}$$

where $p^{\overline{u},f,g,\omega} \in L^2(H_0^1(\Omega)), w^{\overline{u},f,g,\omega} \in L^2(L^2(\Omega))$ and $(y^{\overline{u},f,g,\omega}, v^{\overline{u},f,g,\omega}, \overline{u}, p^{\overline{u},f,g,\omega}, w^{\overline{u},f,g,\omega})$ solves (3.6)–(3.8).

Proof. The optimality system (3.6)–(3.8) can be easily proved using standard techniques, see e.g., [12, Theorem 2.1], [19, Chap. 3]. □

Remark 3.1. Let $U_{ad} \subsetneq U$. Then, instead of (3.8), we find the variational inequality

$$\int_{\Omega\times[0,T]} (\alpha\chi_\omega \overline{u} - \chi_\omega p^{\overline{u},f,g,\omega})(u - \overline{u})\, dxdt \geq 0, \text{ for all } u \in U_{ad}. \tag{3.9}$$

The optimal solution $\overline{u}$ is now characterized using (3.9).

In the following lemma, the well-posedness of J_2 is checked.

Lemma 3.6. *Let K_1 and K_2 be two weakly compact sets containing the respective origins. Then for every $\omega \in E(\Omega)$, we can find $f \in K_1$ and $g \in K_2$ satisfying*

$$\|f\|_{H_0^1(\Omega)} \leq 1,\ \|g\|_{L^2(\Omega)} \leq 1 \text{ and } J_2(\omega) = J_1(\omega, f, g).$$

Proof. Note that $0 \in U_{ad}$. Let $f \in K_1$ and $g \in K_2$ with fixed $\omega \in E(\Omega)$. Then in the absence of control, using (3.3) we have

$$\begin{aligned} J_1(\omega, f, g) = \min_{u\in U_{ad}} J(\omega, u, f, g) &\leq \int_0^T \frac{1}{2}\|y^{0,f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v^{0,f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)}\, dt, \\ &\leq c(\|f\|^2_{H_0^1(\Omega)} + \|g\|^2_{L^2(\Omega)}) \leq cR^2, \end{aligned} \tag{3.10}$$

where $R = \sqrt{2}$. Since $f \in K_1$ and $g \in K_2$, it follows that $\frac{f}{R} \in K_1$ and $\frac{g}{R} \in K_2$ with $\|\frac{f}{R}\|_{H_0^1(\Omega)} \leq 1,\ \|\frac{g}{R}\|_{L^2(\Omega)} \leq 1$. Next, we show that $J_2(\omega) = J_1(\omega, f, g)$. From (2.4), we have

$$\begin{aligned} J_2(\omega) = \sup_{f\in K_1, g\in K_2} \int_0^T &\frac{1}{2}\|y^{\overline{u}^{f,g,\omega},f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v^{\overline{u}^{f,g,\omega},f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)} \\ &+ \frac{\alpha}{2}\|\chi_\omega \overline{u}^{f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)}\, dt. \end{aligned} \tag{3.11}$$

Let $\{f_n\} \subset K_1$, $\|f_n\|_{H_0^1(\Omega)} \leq 1$ and $\{g_n\} \subset K_2$, $\|g_n\|_{L^2(\Omega)} \leq 1$ be maximizing sequences. Then, (3.11) can be written as

$$J_2(\omega) = \lim_{n\to\infty} \int_0^T \frac{1}{2}\|y^{\overline{u}^{f_n,g_n,\omega},f_n,g_n,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v^{\overline{u}^{f_n,g_n,\omega},f_n,g_n,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\chi_\omega \overline{u}^{f_n,g_n,\omega}(\cdot,t)\|^2_{L^2(\Omega)}\, dt. \tag{3.12}$$

Since $\{f_n\}$ and $\{g_n\}$ are bounded in K_1 and K_2, respectively, a subsequence $\{f_{n_k}\}$ converges weakly to $f \in K_1$; $\{g_{n_k}\}$ converges weakly to $g \in K_2$. Since $\{f_n\} \subset K_1$ and $\{g_n\} \subset K_2$, the limit elements satisfy

$$\|f\|_{H_0^1(\Omega)} \leq \lim_{k\to\infty} \inf \|f_{n_k}\|_{H_0^1(\Omega)} \leq 1,\ \|g\|_{L^2(\Omega)} \leq \lim_{k\to\infty} \inf \|g_{n_k}\|_{L^2(\Omega)} \leq 1,$$

by lower semicontinuity of norms. Thus, $\|f\|_{H_0^1(\Omega)} \leq 1$ and $\|g\|_{L^2(\Omega)} \leq 1$. Since $\{f_{n_k}\}$ and $\{g_{n_k}\}$ are bounded in $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively, $f_{n_k} \rightharpoonup f \in H_0^1(\Omega)$ and $g_{n_k} \rightharpoonup g$ in $L^2(\Omega)$. From Lemma 3.2, we note that $\{y^{u_n,f_n,g_n,\omega}\}$ converges strongly to $y^{u,f,g,\omega}$ in $L^2(H_0^1(\Omega))$ and $\{v^{u_n,f_n,g_n,\omega}\}$ converges strongly to $v^{u,f,g,\omega}$ in $L^2(L^2(\Omega))$, and from Lemma 3.4, $\overline{u}^{f_n,g_n,\omega} \to \overline{u}^{f,g,\omega}$ in U_{ad} as $n \to \infty$. Thus, by lower semicontinuity, we have norm convergence. Hence, we may pass to the limit in (3.12) and obtain

$$\begin{aligned} J_2(\omega) &= \frac{1}{2}\int_0^T \|y^{\overline{u}^{f,g,\omega},f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \|v^{\overline{u}^{f,g,\omega},f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \alpha\|\chi_\omega \overline{u}^{f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)} dt \\ &= J_1(\omega,f,g). \end{aligned}$$

Since $f \in K_1$ and $g \in K_2$ satisfy

$$\|f\|_{H_0^1(\Omega)} \leq 1,\ \|g\|_{L^2(\Omega)} \leq 1,\ \max_{f\in K_1, g\in K_2} J_1(\omega,f,g) =: J_2(\omega) = J_1(\omega,f,g),$$

it follows that the map $\omega \mapsto J_2(\omega)$ is well-posed. □

4. Sensitivity Analysis of the Functionals

4.1. **Shape Derivative.** In order to compute the shape derivatives of J_1 and J_2, we introduce a perturbation of the identity. Consider the space $\mathring{C}^{0,1}(\overline{\Omega},\mathbb{R}^2)$ of Lipschitz vector fields vanishing on $\partial\Omega$. We define a perturbation of the identity $\mathbf{T}_\tau(x)$ by $\mathbf{T}_\tau(x) := x + \tau\mathbf{X}(x)$, where $x \in \overline{\Omega}, \mathbf{X} \in \mathring{C}^{0,1}(\overline{\Omega},\mathbb{R}^2)$ and τ is the perturbation parameter [5, p.175]. In view of the perturbation of the identity, we give the definition of a shape derivative of J as follows.

Definition 4.1. The directional derivative of J at $\omega \in E(\Omega)$ in the direction $\mathbf{X} \in \mathring{C}^{0,1}(\overline{\Omega},\mathbb{R}^2)$ is defined by

$$DJ(\omega)(\mathbf{X}) := \lim_{\tau\searrow 0} \frac{J(\mathbf{T}_\tau(\omega)) - J(\omega)}{\tau},$$

provided the limit exists.

Remark 4.1. The cost functional J is shape differentiable at ω if $\mathbf{X} \mapsto DJ(\omega)(\mathbf{X})$ is linear and continuous for all $\mathbf{X} \in \mathring{C}^{0,1}(\overline{\Omega},\mathbb{R}^2)$, see e.g., [5] and [4].

4.2. **Sensitivity of the State Equation.** The space-time cylinder and its boundary will be denoted by $\Omega_T := \Omega \times (0,T]$ and $\Gamma_T := \Gamma \times (0,T]$, respectively. The sensitivity of the solution of (3.1) is given in the following lemma.

Lemma 4.1. *Let $\mathbf{T}_\tau = \mathrm{id} + \tau\mathbf{X}, \tau \geq 0$. Suppose that ω is perturbed such that $\omega_\tau := \mathbf{T}_\tau(\omega), \omega \in E(\Omega)$. Then on the perturbed domain $\Omega_\tau \times (0,T]$ with $\Omega_\tau := \mathbf{T}_\tau(\Omega), \tau \geq 0$, we have*

$$\frac{\partial y^{u,f,g,\tau}}{\partial t} - v^{u,f,g,\tau} = 0 \text{ in } \Omega_T, \tag{4.1}$$

$$\frac{\partial v^{u,f,g,\tau}}{\partial t} - \frac{1}{\zeta(\tau)}\mathrm{div}(A(\tau)\nabla y^{u,f,g,\tau}) = \chi_\omega u \text{ in } \Omega_T, \tag{4.2}$$

$$y^{u,f,g,\tau}(x,0) = f(x) \circ \mathbf{T}_\tau,\ v^{u,f,g,\tau}(x,0) = g(x) \circ \mathbf{T}_\tau \text{ in } \Omega, \tag{4.3}$$

$$y^{u,f,g,\tau} = 0 \text{ on } \Gamma_T, \tag{4.4}$$

where

$$A(\tau) := \zeta(\tau)(\partial\mathbf{T}_\tau)^{-1}(\partial\mathbf{T}_\tau)^{-\top},\ \zeta(\tau) := |\det(\partial\mathbf{T}_\tau)|. \tag{4.5}$$

Proof. In view of (3.1) with $\omega_\tau := \mathbf{T}_\tau(\omega), \omega \in E(\Omega)$, we have

$$\frac{\partial y^{u,f,g,\omega_\tau}}{\partial t} - v^{u,f,g,\omega_\tau} = 0 \text{ in } \Omega_T, \tag{4.6}$$

$$\frac{\partial v^{u,f,g,\omega_\tau}}{\partial t} - \triangle y^{u,f,g,\omega_\tau} = \chi_{\omega_\tau} u \text{ in } \Omega_T, \tag{4.7}$$

$$y^{u,f,g,\omega_\tau}(x,0) = f(x), v^{u,f,g,\omega_\tau}(x,0) = g(x) \text{ in } \Omega, \tag{4.8}$$

$$y^{u,f,g,\omega_\tau} = 0 \text{ on } \Gamma_T, \tag{4.9}$$

where $\omega_\tau \subset \Omega$. Thus, considering (4.7) on the perturbed domain $\Omega_\tau \times (0,T]$ with $\Omega_\tau = \mathbf{T}_\tau(\Omega), \tau \geq 0$, we get the perturbed weak formulation:

$$\int_{\Omega_\tau \times (0,T]} \frac{\partial v^{u,f,g,\omega_\tau}}{\partial t}\varphi\ dx_\tau dt + \int_{\Omega_\tau \times (0,T]} \nabla y^{u,f,g,\omega_\tau} \cdot \nabla\varphi\ dx_\tau dt = \int_{\Omega_\tau \times (0,T]} \chi_{\omega_\tau} u\varphi\ dx_\tau\ dt, \tag{4.10}$$

for all $\varphi \in L^2(H_0^1(\Omega_\tau))$ with $(y^{u,f,g,\omega_\tau}, v^{u,f,g,\omega_\tau})$ satisfying (4.6)–(4.9). Next, employing a change of variables induced by $\Omega_\tau := \mathbf{T}_\tau(\Omega)$ in (4.10) gives

$$\begin{aligned}&\int_{\Omega_T} \zeta(\tau)\frac{\partial(v^{u\circ\mathbf{T}_\tau^{-1},f,g,\omega_\tau} \circ \mathbf{T}_\tau)}{\partial t}(\varphi \circ \mathbf{T}_\tau)\ dxdt + \int_{\Omega_T} \zeta(\tau)\nabla(y^{u\circ\mathbf{T}_\tau^{-1},f,g,\omega_\tau} \circ \mathbf{T}_\tau) \cdot \nabla(\varphi \circ \mathbf{T}_\tau)\ dxdt \\ &= \int_{\Omega_T} \zeta(\tau)(\chi_{\omega_\tau} u \circ \mathbf{T}_\tau)(\varphi \circ \mathbf{T}_\tau)\ dxdt,\ \text{for all } \varphi \in L^2(H_0^1(\Omega_\tau)).\end{aligned} \tag{4.11}$$

Applying the chain rule (see e.g., [17, p.63]) in (4.11) together with $\chi_{\omega_\tau} = \chi_\omega \circ \mathbf{T}_\tau^{-1}$ and the perturbed variables (see e.g., [5, p.523])

$$y^{u,f,g,\tau} = y^{u\circ\mathbf{T}_\tau^{-1},f,g,\omega_\tau} \circ \mathbf{T}_\tau,\ v^{u,f,g,\tau} = v^{u\circ\mathbf{T}_\tau^{-1},f,g,\omega_\tau} \circ \mathbf{T}_\tau, \tag{4.12}$$

yield

$$\begin{aligned}&\int_{\Omega_T} \zeta(\tau)\frac{\partial v^{u,f,g,\tau}}{\partial t}(\varphi \circ \mathbf{T}_\tau) + \zeta(\tau)(\partial\mathbf{T}_\tau)^{-\top}\nabla y^{u,f,g,\tau} \cdot (\partial\mathbf{T}_\tau)^{-\top}\nabla(\varphi \circ \mathbf{T}_\tau)\ dxdt \\ &= \int_{\Omega_T} \zeta(\tau)(\chi_\omega u)(\varphi \circ \mathbf{T}_\tau)\ dxdt,\ \text{for all } \varphi \in L^2(H_0^1(\Omega_\tau)).\end{aligned} \tag{4.13}$$

From (4.5), equality (4.13) simplifies to

$$\begin{aligned}&\int_{\Omega_T} \zeta(\tau)\frac{\partial v^{u,f,g,\tau}}{\partial t}(\varphi \circ \mathbf{T}_\tau)\ dxdt + \int_{\Omega_T} A(\tau)\nabla y^{u,f,g,\tau} \cdot \nabla(\varphi \circ \mathbf{T}_\tau)\ dxdt \\ &= \int_{\Omega_T} \zeta(\tau)(\chi_\omega u)(\varphi \circ \mathbf{T}_\tau)\ dxdt,\ \text{for all } \varphi \in L^2(H_0^1(\Omega_\tau)).\end{aligned} \tag{4.14}$$

Since (4.14) is true for all $\varphi \in L^2(H_0^1(\Omega_\tau))$, it follows that for all $\phi \in L^2(H_0^1(\Omega))$ the function $\phi \circ \mathbf{T}_\tau^{-1}$ belongs to $L^2(H_0^1(\Omega_\tau))$. So, testing (4.14) with $\varphi := \phi \circ \mathbf{T}_\tau^{-1}$ for an arbitrary $\phi \in L^2(H_0^1(\Omega))$, we obtain

$$\begin{aligned}&\int_{\Omega_T} \zeta(\tau)\frac{\partial v^{u,f,g,\tau}}{\partial t}(\phi \circ \mathbf{T}_\tau^{-1} \circ \mathbf{T}_\tau) + A(\tau)\nabla y^{u,f,g,\tau} \cdot \nabla(\phi \circ \mathbf{T}_\tau^{-1} \circ \mathbf{T}_\tau)\, dxdt \\ &= \int_{\Omega_T} \zeta(\tau)(\chi_\omega u)(\phi \circ \mathbf{T}_\tau^{-1} \circ \mathbf{T}_\tau)\, dxdt, \text{ for all } \phi \in L^2(H_0^1(\Omega)).\end{aligned} \tag{4.15}$$

Rewriting (4.15), we have

$$\begin{aligned}&\int_{\Omega_T} \zeta(\tau)\frac{\partial v^{u,f,g,\tau}}{\partial t}\phi\, dxdt + \int_{\Omega_T} A(\tau)\nabla y^{u,f,g,\tau} \cdot \nabla\phi\, dxdt \\ &= \int_{\Omega_T} \zeta(\tau)\chi_\omega u\phi\, dxdt, \text{ for all } \phi \in L^2(H_0^1(\Omega)).\end{aligned} \tag{4.16}$$

Similarly, considering (4.6) on $\Omega_\tau \times (0,T]$, it can be shown that:

$$\int_{\Omega_T} \zeta(\tau)\frac{\partial y^{u,f,g,\tau}}{\partial t}\psi - \zeta(\tau)v^{u,f,g,\tau}\psi\, dxdt = 0, \text{ for all } \psi \in L^2(L^2(\Omega)). \tag{4.17}$$

Thus, after mapping back (4.16) and (4.17), and using (4.8)–(4.9) in (4.12), we have (4.1)–(4.4). $\square$

In the following essential lemma, the sequence $\{\tau_n\}_{n=1}^{\infty}$ will be necessary.

Lemma 4.2. *Let* $\mathbf{X} \in \mathring{C}^{0,1}(\overline{\Omega}, \mathbb{R}^2)$.

(a) *Then as* $\tau_n \to 0^+$, *we have*

$$\frac{\zeta(\tau_n) - 1}{\tau_n} \to \operatorname{div}(\mathbf{X}) \text{ strongly in } L^\infty(\overline{\Omega}), \tag{4.18}$$

$$\frac{A(\tau_n) - I}{\tau_n} \to \operatorname{div}(\mathbf{X})I - \partial\mathbf{X} - \partial\mathbf{X}^\top \text{ strongly in } L^\infty(\overline{\Omega}, \mathbb{R}^{2\times 2}), \tag{4.19}$$

where I *is the* 2*-dimensional identity matrix.*

(b) *Suppose that* $\{\Psi_n\}$ *is a sequence in* $H_0^1(\Omega)$ *converging weakly to* $\Psi \in H_0^1(\Omega)$.

  (i) *Then for all* $\Psi \in H_0^1(\Omega)$, *we have as* $\tau \to 0^+$,

$$\Psi_n \circ \mathbf{T}_\tau \to \Psi \text{ strongly in } H_0^1(\Omega). \tag{4.20}$$

  (ii) *If* $\{\tau_n\}$ *is a null sequence, then as* $n \to \infty$ *we have*

$$\frac{\Psi_n \circ \mathbf{T}_{\tau_n} - \Psi_n}{\tau_n} \rightharpoonup \nabla\Psi \cdot \mathbf{X} \text{ weakly in } H_0^1(\Omega). \tag{4.21}$$

Proof. The results of the convergence (4.18), (4.19), and (4.21) are proved in [17]: Lemma 2.31, p.107 and proposition 2.72, respectively while (4.20) is proved in [5, p.527]. $\square$

Remark 4.2. There are constants $c_1, c_2 > 0$ such that for all $x \in \Omega$ and $\tau \in [0, \tau_{\mathbf{X}}], \tau_{\mathbf{X}} \geq 0$,

$$c_1 \leq \zeta(\tau)(x),\ c_2|\zeta|^2 \leq A(\tau)(x)\zeta \cdot \zeta, \tag{4.22}$$

for all $\zeta \in \mathbb{R}^2$, see e.g., [5, p.559].

The following lemma gives the a-priori estimates for y^{u,f,g,ω_τ}, $y^{u,f,g,\tau}$, v^{u,f,g,ω_τ} and $v^{u,f,g,\tau}$.

Lemma 4.3. *For all* $(u, f, g, \omega) \in U_{ad} \times H_0^1(\Omega) \times L^2(\Omega) \times E(\Omega)$, *there exists a constant* $c > 0$, *such that*

$$\|y^{u,f,g,\omega_\tau}\|_{L^2(H_0^1(\Omega))} + \|v^{u,f,g,\omega_\tau}\|_{L^2(L^2(\Omega))} \leq c\Big(\|\chi_{\omega_\tau}u\|_{L^2(L^2(\Omega))} + \|f\|_{H_0^1(\Omega)} + \|g\|_{L^2(\Omega)}\Big), \tag{4.23}$$

$$\|y^{u,f,g,\tau}\|_{L^2(H_0^1(\Omega))} + \|v^{u,f,g,\tau}\|_{L^2(L^2(\Omega))} \le c\Big(\|\chi_\omega u\|_{L^2(L^2(\Omega))} + \|f\|_{H_0^1(\Omega)} + \|g\|_{L^2(\Omega)}\Big). \tag{4.24}$$

Proof. Note that (4.23) is a consequence of (3.3) and the proof is omitted here. We prove (4.24) as follows. By a change of variables, we have

$$\begin{aligned}
&\int_0^T \|y^{u,f,g,\tau}(\cdot,t)\|^2_{L^2(\Omega)} + \|\nabla y^{u,f,g,\tau}(\cdot,t)\|^2_{L^2(\Omega)}\, dt \\
&= \int_{\Omega_T} |y^{u,f,g,\tau}|^2 + |\nabla y^{u,f,g,\tau}|^2\, dxdt, \\
&= \int_{\Omega_T} \zeta^{-1}(\tau)|y^{u,f,g,\tau}\circ T_\tau^{-1}|^2 + \zeta^{-1}(\tau)\nabla y^{u,f,g,\tau}\circ T_\tau^{-1}\cdot \nabla y^{u,f,g,\tau}\circ T_\tau^{-1}\, dxdt, \qquad (4.25)\\
&= \int_{\Omega_T} \zeta^{-1}(\tau)|y^{u\circ \mathbf{T}_\tau^{-1},f,g,\omega_\tau}|^2 + A^{-1}(\tau)\nabla y^{u\circ \mathbf{T}_\tau^{-1},f,g,\omega_\tau}\cdot \nabla y^{u\circ \mathbf{T}_\tau^{-1},f,g,\omega_\tau}\, dxdt, \\
&\overset{(4.22)}{\le} c\int_{\Omega_T} |y^{u\circ \mathbf{T}_\tau^{-1},f,g,\omega_\tau}|^2 + \nabla y^{u\circ \mathbf{T}_\tau^{-1},f,g,\omega_\tau}\cdot \nabla y^{u\circ \mathbf{T}_\tau^{-1},f,g,\omega_\tau}\, dxdt, \\
&\overset{(4.23)}{\le} c\Big(\|\chi_{\omega_\tau} u\circ \mathbf{T}_\tau^{-1}\|^2_{L^2(L^2(\Omega))} + \|f\|^2_{H_0^1(\Omega)} + \|g\|^2_{L^2(\Omega)}\Big).
\end{aligned}$$

Using $\chi_{\omega_\tau} = \chi_\omega \circ T_\tau^{-1}$ and the natural norm on $H^1(\Omega)$, i.e.,

$$\int_0^T \|y^{u,f,g,\tau}(\cdot,t)\|^2_{L^2(\Omega)} + \|\nabla y^{u,f,g,\tau}(\cdot,t)\|^2_{L^2(\Omega)}\, dt = \|y^{u,f,g,\tau}\|^2_{L^2(H^1(\Omega))},$$

in (4.25) (see e.g., [3, p.39]), we obtain the desired inequality. □

For the continuity results of $(u,f,g,\tau)\mapsto y^{u,f,g,\tau}$ and $(u,f,g,\tau)\mapsto v^{u,f,g,\tau}$, we prove the lemma that follows.

Lemma 4.4. *For every $(\omega_1,u_1,f_1,g_1),(\omega_2,u_2,f_2,g_2)\in E(\Omega)\times U_{ad}\times H_0^1(\Omega)\times L^2(\Omega)$, with (y_1,v_1) and (y_2,v_2) being the corresponding solutions to (4.6)–(4.9), there is a constant $c>0$, independent of (ω_1,u_1,f_1,g_1) and (ω_2,u_2,f_2,g_2), such that*

$$\begin{aligned}
&\|y_1-y_2\|_{L^2(H_0^1(\Omega))} + \|v_1-v_2\|_{L^2(L^2(\Omega))}\\
&\le c\Big(\|\chi_{\omega_1}u_1 - \chi_{\omega_2}u_2\|_{L^2(L^2(\Omega))} + \|f_1-f_2\|_{H_0^1(\Omega)} + \|g_1-g_2\|_{L^2(\Omega)}\Big).
\end{aligned} \tag{4.26}$$

Proof. Since (y_1,v_1) and (y_2,v_2) solve (4.6)–(4.9), it follows that they satisfy

$$\begin{aligned}
\frac{\partial y_k}{\partial t} - v_k &= 0 \text{ in } \Omega_T,\\
\frac{\partial v_k}{\partial t} - \triangle y_k &= \chi_{\omega_k}u_k \text{ in } \Omega_T,\\
y_k(x,0) = f_k(x), v_k(x,0) &= g_k(x) \text{ in } \Omega,\\
y_k &= 0 \text{ on } \Gamma_T,
\end{aligned}$$

for all $k=1,2$. Let $y_{12} := y_1 - y_2$ and $v_{12} := v_1 - v_2$. Then (y_{12},v_{12}) satisfies

$$\begin{aligned}
\frac{\partial y_{12}}{\partial t} - v_{12} &= 0 \text{ in } \Omega_T,\\
\frac{\partial v_{12}}{\partial t} - \triangle y_{12} &= \chi_{\omega_1}u_1 - \chi_{\omega_2}u_2 \text{ in } \Omega_T,\\
y_{12}(x,0) = f_1(x)-f_2(x), v_{12}(x,0) &= g_1(x)-g_2(x) \text{ in } \Omega,\\
y_{12} &= 0 \text{ on } \Gamma_T.
\end{aligned}$$

Hence, (4.26) follows from (3.3). $\square$

The following lemma is an immediate consequence of Lemma 4.4.

Lemma 4.5. *Let $\omega \in E(\Omega)$ be given. Suppose that for all $\tau_n \in (0, \tau_{\mathbf{X}}]$, $u_n, u \in U_{ad}$, $f_n, f \in H_0^1(\Omega)$ and $g_n, g \in L^2(\Omega)$,*

$$u_n \rightharpoonup u \text{ in } U_{ad},\ f_n \rightharpoonup f \text{ in } H_0^1(\Omega), g_n \rightharpoonup g \text{ in } L^2(\Omega),\ \tau_n \to 0,\ \text{as } n \to \infty.$$

Then:

$$\begin{aligned} y^{u_n,f_n,g_n,\tau_n} &\to y^{u,f,g,\omega} \text{ in } L^2(H_0^1(\Omega)) \text{ as } n \to \infty, \\ v^{u_n,f_n,g_n,\tau_n} &\to v^{u,f,g,\omega} \text{ in } L^2(L^2(\Omega)) \text{ as } n \to \infty. \end{aligned}$$

Proof. Using inequality (4.24), we see that the sequences $\{y^{u_n,f_n,g_n,\tau_n}\}$ and $\{v^{u_n,f_n,g_n,\tau_n}\}$ are bounded in $L^2(H^2(\Omega) \cap H_0^1(\Omega))$ and $L^2(H_0^1(\Omega) \cap L^2(\Omega))$, respectively. By Rellich-Kondrachov theorem, we can extract subsequences $\{y^{u_{n_k},f_{n_k},g_{n_k},\tau_{n_k}}\}$ and $\{v^{u_{n_k},f_{n_k},g_{n_k},\tau_{n_k}}\}$ such that $\{y^{u_{n_k},f_{n_k},g_{n_k},\tau_{n_k}}\}$ converges weakly to $y^{u,f,g,\omega}$ in $L^2(H^2(\Omega) \cap H_0^1(\Omega))$ and strongly to $y^{u,f,g,\omega}$ in $L^2(H_0^1(\Omega))$, and $\{v^{u_{n_k},f_{n_k},g_{n_k},\tau_{n_k}}\}$ converges weakly to $v^{u,f,g,\omega}$ in $L^2(H_0^1(\Omega))$ and strongly to $v^{u,f,g,\omega}$ in $L^2(L^2(\Omega))$. From (4.16) and (4.17), it is known that (y_k, v_k) with $y_k := y^{u_{n_k},f_{n_k},g_{n_k},\tau_{n_k}}$ and $v_k := v^{u_{n_k},f_{n_k},g_{n_k},\tau_{n_k}}$, $k \in \{0\} \cup \mathbb{N}$ satisfies the variational formulations

$$\begin{aligned} &\int_{\Omega_T} \zeta(\tau_{n_k}) \frac{\partial v_k}{\partial t} \varphi + A(\tau_{n_k}) \nabla y_k \cdot \nabla \varphi \ dxdt = \int_{\Omega_T} \zeta(\tau_{n_k}) \chi_\omega u_{n_k} \varphi \ dxdt, \\ &\int_{\Omega_T} \zeta(\tau_{n_k}) \frac{\partial y_k}{\partial t} \psi \ dxdt - \int_{\Omega_T} \zeta(\tau_{n_k}) v_k \psi \ dxdt = 0, \end{aligned} \tag{4.27}$$

for all $\varphi \in L^2(H_0^1(\Omega))$ and $\psi \in L^2(L^2(\Omega))$ with $y_k(x,0) = f_{n_k}(x) \circ T_{\tau_{n_k}}$ and $v_k(x,0) = g_{n_k}(x) \circ T_{\tau_{n_k}}$ in Ω. From Lemma 4.2, it follows that $f_{n_k}(x) \circ \mathbf{T}_{\tau_{n_k}} \to f(x)$ in $H_0^1(\Omega)$ and $g_{n_k}(x) \circ \mathbf{T}_{\tau_{n_k}} \to g(x)$ in $L^2(\Omega)$ as $k \to \infty$. Thus, we have $y(x,0) = f(x)$ and $v(x,0) = g(x)$. Using the weak convergence of $\{u_{n_k}\}$, $\{y_k\}$, $\{v_k\}$ and the strong convergence in Lemma 4.2, i.e., $\zeta(\tau_{n_k}) \to 1$ in $L^\infty(\Omega)$, $A(\tau_{n_k}) \to I$ in $L^\infty(\Omega, \mathbb{R}^{2\times 2})$ as $k \to \infty$, we pass to the limits in (4.27) and obtain

$$\begin{aligned} &\int_{\Omega_T} \frac{\partial v}{\partial t} \varphi + \nabla y \cdot \nabla \varphi \ dxdt = \int_{\Omega_T} \chi_\omega u \varphi \ dxdt, \\ &\int_{\Omega_T} \frac{\partial y}{\partial t} \psi \ dxdt - \int_{\Omega_T} v \psi \ dxdt = 0, \end{aligned} \tag{4.28}$$

for all $\varphi \in L^2(H_0^1(\Omega))$ and $\psi \in L^2(L^2(\Omega))$ with $y(x,0) = f(x), v(x,0) = g(x)$. Furthermore, since (4.28) with $y(x,0) = f(x), v(x,0) = g(x)$ admits a unique solution, we must have $y = y^{u,f,g,\omega}$ and $v = v^{u,f,g,\omega}$. Thus, the sequences $\{y_n\}$ and $\{v_n\}$ converge to $y = y^{u,f,g,\omega}$ in $L^2(H_0^1(\Omega))$ and $v = v^{u,f,g,\omega}$ in $L^2(L^2(\Omega))$, respectively. This finishes the proof. $\square$

The following lemmas will be employed in the proof of the theorem that follows.

Lemma 4.6. *For every null-sequence $\{\tau_n\}$ in $[0, \tau_{\mathbf{X}}]$, every sequence $\{f_n\}$ in K_1 converging weakly in $H_0^1(\Omega)$ to $f \in K_1$ and for every sequence $\{g_n\}$ in K_2 converging weakly in $L^2(\Omega)$ to $g \in K_2$, we have*

$$\overline{u}^{f_n,g_n,\tau_n} \to \overline{u}^{f,g,\omega} \text{ in } U_{ad} \text{ as } n \to \infty.$$

Proof. We proceed as follows. Note that ω_{τ_n}, $\overline{u}^{f_n,g_n,\omega_{\tau_n}}$ and $\overline{u}^{f_n,g_n,\tau_n}$ represent the perturbed domain, optimal control solution, and perturbed optimal control, respectively. Since $\overline{u}^{f_n,g_n,\tau_n} = \overline{u}^{f_n,g_n,\omega_{\tau_n}} \circ \mathbf{T}_{\tau_n}$ (see e.g., [5, p.523]) and $\overline{u}^{f_n,g_n,\omega_{\tau_n}} \to \overline{u}^{f,g,\omega}$ in U_{ad} by Lemma 3.4, the desired result follows from Lemma 4.2. $\square$

In the sequel, we denote the set of maximizers by $\mathfrak{X}_2(\omega)$.

Lemma 4.7. *For every null sequence $\{\tau_n\}$ in $[0, \tau_{\mathbf{X}}]$ and every sequence $\{f_n, g_n\}$ with $(f_n, g_n) \in \mathfrak{X}_2(\omega_{\tau_n})$, we can find a subsequence $\{f_{n_k}, g_{n_k}\}$, such that $f_{n_k} \rightharpoonup f$ in $H_0^1(\Omega)$ and $g_{n_k} \rightharpoonup g$ in $L^2(\Omega)$ as $k \to \infty$, where $(f, g) \in \mathfrak{X}_2(\omega)$.*

Proof. It is easy to prove this from (2.4) and the proof is left out. □

4.3. **Averaged Adjoint Equations.** Let $\tau \in [0, \tau_{\mathbf{X}}]$ be fixed. Then the mapping $\mathbf{T}_\tau^{-1} : U_{ad} \to U_{ad},\ u \mapsto \mathbf{T}_\tau^{-1} \circ u$ is a bijection between U_{ad} and U_{ad} that preserves the binary operations. As a consequence and using the change of variables $\mathbf{T}_\tau$, it is easy to show that

$$\inf_{u \in U_{ad}} J(\omega_\tau, u, f, g) = \frac{1}{2} \inf_{u \in U_{ad}} \int_{\Omega_T} \zeta(\tau) \Big(|y^{u,f,g,\tau}|^2 + |v^{u,f,g,\tau}|^2 + \alpha |u|^2 \Big)\, dxdt.$$

Note that $p \in L^2(H_0^1(\Omega))$ and $w \in L^2(L^2(\Omega))$. By choosing Lagrange multipliers $\phi = p$ and $\psi = w$, we can incorporate (4.1)–(4.4) in the formulation of the following Lagrangian functional.

Definition 4.2. Define the parametrized Lagrangian
$\tilde{H} : [0, \tau_{\mathbf{X}}] \times U_{ad} \times K_1 \times K_2 \times H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ by

$$\begin{aligned}
H(\tau, u, f, g) &:= \int_{\Omega_T} \frac{1}{2} \zeta(\tau) \Big((y^{u,f,g,\tau})^2 + (v^{u,f,g,\tau})^2 + \alpha (u)^2 \Big)\, dxdt \\
&+ \int_{\Omega_T} \zeta(\tau) \frac{\partial v^{u,f,g,\tau}}{\partial t} p^{u,f,g,\tau} + A(\tau) \nabla y^{u,f,g,\tau} \cdot \nabla p^{u,f,g,\tau} \ - \zeta(\tau) \chi_\omega u p^{u,f,g,\tau} \\
&+ \zeta(\tau) \frac{\partial y^{u,f,g,\tau}}{\partial t} w^{u,f,g,\tau} - \zeta(\tau) v^{u,f,g,\tau} w^{u,f,g,\tau}\ dxdt \\
&+ \int_{\Omega} \zeta(\tau)(y^{u,f,g,\tau}(x,0) - f \circ \mathbf{T}_\tau) w^{u,f,g,\tau}(x,0) + \zeta(\tau)(v^{u,f,g,\tau}(x,0) - g \circ \mathbf{T}_\tau) p^{u,f,g,\tau}(x,0) dx,
\end{aligned} \tag{4.29}$$

where $H(\tau, u, f, g) := \tilde{H}(\tau, u, f, g, y^{u,f,g,\tau}, v^{u,f,g,\tau}, p^{u,f,g,\tau}, w^{u,f,g,\tau})$.

In the sequel, the following definition is used.

Definition 4.3. Given $\tau \in [0, \tau_{\mathbf{X}}]$, $0 \le s \le 1$ and $(u, f, g) \in U_{ad} \times K_1 \times K_2$. We define the averaged adjoint equations associated with $y^{u,f,g,\tau}$ and $y^{u,f,g,\omega}$; $v^{u,f,g,\tau}$ and $v^{u,f,g,\omega}$ as: find $p^{u,f,g,\tau} \in L^2(H_0^1(\Omega))$ and $w^{u,f,g,\tau} \in L^2(L^2(\Omega))$ such that

$$\int_0^1 \partial_y \tilde{H}(\tau, u, f, g, s y^{u,f,g,\tau} + (1-s) y^{u,f,g,\omega}, v^{u,f,g,\tau}, p^{u,f,g,\tau}, w^{u,f,g,\tau})(\phi) ds = 0, \tag{4.30}$$

for all $\phi \in L^2(H_0^1(\Omega))$, and

$$\int_0^1 \partial_v \tilde{H}(\tau, u, f, g, y^{u,f,g,\tau}, s v^{u,f,g,y,\tau} + (1-s) v^{u,f,g,\omega}, p^{u,f,g,\tau}, w^{u,f,g,\tau})(\psi) ds = 0, \tag{4.31}$$

for all $\psi \in L^2(L^2(\Omega))$, where $\partial_y \tilde{H}$ and $\partial_v \tilde{H}$ denote the partial derivatives of $\tilde{H}$ with respect to y and v, respectively.

The following lemmas will be important in the proof of the theorem that follows.

Lemma 4.8. *The averaged adjoint equations (4.30) and (4.31), associated with $y^{u,f,g,\tau}$ and $y^{u,f,g,\omega}$; $v^{u,f,g,\tau}$ and $v^{u,f,g,\omega}$ are given by*

$$\begin{aligned}&\int_{\Omega_T} -\zeta(\tau)\phi\frac{\partial w^{u,f,g,\tau}}{\partial t}\ dxdt + \int_{\Omega_T} A(\tau)\nabla\phi\cdot\nabla p^{u,f,g,\tau}\ dxdt\\ &= -\int_{\Omega_T}\frac{1}{2}\zeta(\tau)(y^{u,f,g,\tau}+y^{u,f,g,\omega})\phi\ dxdt,\ \text{for all } \phi\in L^2(H_0^1(\Omega))\end{aligned}\tag{4.32}$$

and

$$\int_{\Omega_T} -\zeta(\tau)\psi\left(\frac{\partial p^{u,f,g,\tau}}{\partial t}+w^{u,f,g,\tau}\right)\ dxdt = -\int_{\Omega_T}\frac{1}{2}\zeta(\tau)(v^{u,f,g,\tau}+v^{u,f,g,\omega})\psi\ dxdt,\tag{4.33}$$

for all $\psi\in L^2(L^2(\Omega))$, respectively.

Proof. Since $y^{u,f,g,\tau}\mapsto \tilde{H}(\tau,u,f,g,y^{u,f,g,\tau},v^{u,f,g,\tau},p^{u,f,g,\tau},w^{u,f,g,\tau})$ is affine, $\tilde{H}$ is Gâteaux differentiable with respect to y, see e.g., [19, p.200]. Thus, it is easy to see that (4.32) and (4.33) hold. □

The lemma that follows is a direct consequence of Lemmas 4.5 and 4.8.

Lemma 4.9. *For all $\tau_n\in(0,\tau_{\mathbf{X}}]$, $u_n\in U_{ad}$, $f_n\in K_1$ and $g_n\in K_2$, such that*

$$u_n\rightharpoonup u \text{ in } U_{ad},\ f_n\rightharpoonup f \text{ in } H_0^1(\Omega), g_n\rightharpoonup g \text{ in } L^2(\Omega),\ \tau_n\to 0,\ \text{as } n\to\infty,$$

where $u\in U_{ad}$, $f\in K_1$ and $g\in K_2$, we have

$$\begin{aligned}p^{u_n,f_n,g_n,\tau_n}&\to p^{u,f,g,\omega} \text{ in } L^2(H_0^1(\Omega)) \text{ as } n\to\infty,\\ w^{u_n,f_n,g_n,\tau_n}&\to w^{u,f,g,\omega} \text{ in } L^2(L^2(\Omega)) \text{ as } n\to\infty,\end{aligned}$$

with $p^{u,f,g,\omega}\in L^2(H_0^1(\Omega))$ and $w^{u,f,g,\omega}\in L^2(L^2(\Omega))$ satisfying the adjoint equations

$$\begin{aligned}&\int_{\Omega_T} -\phi\frac{\partial w^{u,f,g,\omega}}{\partial t}\ dxdt + \int_{\Omega_T}\nabla\phi\cdot\nabla p^{u,f,g,\omega}\ dxdt = -\int_{\Omega_T} y^{u,f,g,\omega}\phi\ dxdt,\\ &\int_{\Omega_T} -\psi\frac{\partial p^{u,f,g,\omega}}{\partial t} - \psi w^{u,f,g,\omega}\ dxdt = -\int_{\Omega_T} v^{u,f,g,\omega}\psi\ dxdt,\end{aligned}$$

for all $\phi\in L^2(H_0^1(\Omega))$ and $\psi\in L^2(L^2(\Omega))$ with $p^{u,f,g,\omega}(x,T)=0$ and $w^{u,f,g,\omega}(x,T)=0$ a.e. in Ω.

Proof. Using (3.6)–(3.7) and the estimate in [6, p.391-393, Theorem 6] , we have the a-priori bound for the adjoint given by

$$\|p^{u,f,g,\omega}\|_{L^2(H_0^1(\Omega))} + \left\|\frac{\partial p^{u,f,g,\omega}}{\partial t}\right\|_{L^2(L^2(\Omega))} \le c\|v^{u,f,g,\tau}+v^{u,f,g,\omega}\|_{L^2(L^2(\Omega))}.\tag{4.34}$$

Using similar arguments as in Lemma 4.5 and replacing (u,f,g,τ) by (u_n,f_n,g_n,τ_n) in (4.32) and (4.33), and passing to the limits as $n\to\infty$, we have the desired result. □

4.4. **Directional Derivative of Max-Min Functions.** Let $H:[0,\tau_{\mathbf{X}}]\times U_{ad}\times K_1\times K_2\to\mathbb{R}$ be a function. Then, we define the max-min function $h:[0,\tau_{\mathbf{X}}]\to\mathbb{R}$ by

$$h(\tau) := \sup_{f\in K_1,g\in K_2}\inf_{u\in U_{ad}} H(\tau,u,f,g).$$

In the following lemma, we seek to find out sufficient conditions for the existence of the limit

$$\frac{d}{d\ell}h(0^+) := \lim_{\tau\searrow 0^+}\frac{h(\tau)-h(0)}{\ell(0)},$$

for any function $\ell:[0,\tau_{\mathbf{X}}]\to\mathbb{R}$ such that $\ell(\tau)>0$ for $\tau\in(0,\tau_{\mathbf{X}}]$, and $\ell(0)=0$.

Lemma 4.10. *Assume that the following hypotheses hold.*

(H0) *The problem*

$$\inf_{u\in U_{ad}} H(\tau,u,f,g)$$

admits a unique optimal solution $\overline{u}$.

(H1) *The set of maximizers*

$$\mathfrak{X}_2(\omega) := \{(f,g) : \sup_{f\in K_1, g\in K_2} \inf_{u\in U_{ad}} H(\tau,u,f,g) = \inf_{u\in U_{ad}} H(\tau,u^{\tau,f,g},f,g)\}$$

is nonempty for all $\tau\in[0,\tau_{\mathbf{X}}]$.

(H2) *For all* $f\in K_1, g\in K_2$ *and* $\tau\in[0,\tau_{\mathbf{X}}]$, *the partial derivatives*

$$\lim_{\tau\searrow 0}\frac{H(\tau,u^{\tau,f,g},f,g)-H(0,u^{\tau,f,g},f,g)}{\ell(\tau)}$$

and

$$\lim_{\tau\searrow 0}\frac{H(\tau,u^{0,f,g},f,g)-H(0,u^{0,f,g},f,g)}{\ell(\tau)}$$

exist and are equal.

(H3) *For all* $\tau_n\in[0,\tau_{\mathbf{X}}]$ *and* $(f_n,g_n)\in\mathfrak{X}_2(\omega_n)$, *there exist subsequences* $\{\tau_{n_k}\}$ *and* $\{f_{n_k},g_{n_k}\}$ *with* $f_{n_k}\rightharpoonup f$ *in* $H_0^1(\Omega)$ *and* $g_{n_k}\rightharpoonup g$ *in* $L^2(\Omega)$ *as* $k\to\infty$ *and* $(f,g)\in\mathfrak{X}_2(\omega)$, *such that*

$$\lim_{k\to\infty}\frac{H(\tau_{n_k},u_{n_k},f_{n_k},g_{n_k})-H(0,u_{n_k},f_{n_k},g_{n_k})}{\ell(\tau_{n_k})} = \partial_\ell H(0^+,u^{0,f,g},f,g)$$

and

$$\lim_{k\to\infty}\frac{H(\tau_{n_k},u^{f_{n_k},g_{n_k},0},f_{n_k},g_{n_k})-H(0,u^{f_{n_k},g_{n_k},0},f_{n_k},g_{n_k})}{\ell(\tau_{n_k})} = \partial_\ell H(0^+,u^{f,g,0},f,g).$$

Then, we have

$$\frac{d}{d\ell}h(\tau)|_{\tau=0^+} = \max_{(f,g)\in\mathfrak{X}_2(\omega)}\partial_\ell H(0^+,u^{0,f,g},f,g).$$

Proof. We refer to [5, p.524] and [18]. □

In the following theorem, we derive the directional derivative of J_2 for $\ell(\tau)=\tau$.

Theorem 4.11. *The directional derivative of* $J_2(\omega)$ *at* ω *in the direction* $\mathbf{X}\in\mathring{C}^{0,1}(\overline{\Omega},\mathbb{R}^2)$ *is given by*

$$DJ_2(\omega)(\mathbf{X}) = \max_{(f,g)\in\mathfrak{X}_2(\omega)}\int_{\Omega_T} S_1\Big(\overline{y}^{f,g,\omega},\overline{v}^{f,g,\omega},\overline{p}^{f,g,\omega},\overline{w}^{f,g,\omega},\overline{u}^{f,g,\omega}\Big) : \partial\mathbf{X} + S_0(f,g)\cdot\mathbf{X}\,dxdt, \tag{4.35}$$

where

$$\begin{aligned}
&S_1\Big(\overline{y}^{f,g,\omega},\overline{v}^{f,g,\omega},\overline{p}^{f,g,\omega},\overline{w}^{f,g,\omega},\overline{u}^{f,g,\omega}\Big)\\
&:= \Big(\frac{1}{2}|\overline{y}^{f,g,\omega}|^2+\frac{1}{2}|\overline{v}^{f,g,\omega}|^2+\frac{\alpha}{2}|\overline{u}^{f,g,\omega}|^2-\overline{v}^{f,g,\omega}\frac{\partial\overline{p}^{f,g,\omega}}{\partial t}\\
&-\overline{y}^{f,g,\omega}\frac{\partial\overline{w}^{f,g,\omega}}{\partial t}+\nabla\overline{y}^{f,g,\omega}\cdot\nabla\overline{p}^{f,g,\omega}-\chi_\omega\overline{u}^{f,g,\omega}\overline{p}^{f,g,\omega}-\overline{v}^{f,g,\omega}\overline{w}^{f,g,\omega}\\
&-\frac{1}{T}g\overline{p}^{f,g,\omega}(x,0)-\frac{1}{T}f\overline{w}^{f,g,\omega}(x,0)\Big)I-\nabla\overline{y}^{f,g,\omega}\otimes\nabla\overline{p}^{f,g,\omega}-\nabla\overline{p}^{f,g,\omega}\otimes\nabla\overline{y}^{f,g,\omega},
\end{aligned}$$

$$S_0(f,g) := -\frac{1}{T}\Big(\nabla f\overline{w}^{f,g,\omega}(x,0)+\nabla g\overline{p}^{f,g,\omega}(x,0)\Big), \tag{4.36}$$

and the adjoint $(\overline{p}^{f,g,\omega},\overline{w}^{f,g,\omega})$ *satisfies* (3.6)–(3.7).

Proof. Since J_1 and J_2 are well-posed, it follows that (H0) and (H1) are satisfied. Next, we check that (H2) and (H3) hold. Using the fundamental theorem of calculus on averaged adjoint equations (4.30)–(4.31), it is easy to see that

$$\tilde{H}(\tau, u, f, g, y^{u,f,g,\tau}, v^{u,f,g,\tau}, p^{u,f,g,\tau}, w^{u,f,g,\tau}) = \tilde{H}(\tau, u, f, g, y^{u,f,g,\omega}, v^{u,f,g,\omega}, p^{u,f,g,\tau}, w^{u,f,g,\tau}). \quad (4.37)$$

Since $J(\omega_\tau, u \circ \mathbf{T}_\tau^{-1}, f, g) = \tilde{H}(\tau, u, f, g, y^{u,f,g,\tau}, v^{u,f,g,\tau}, p^{u,f,g,\tau}, w^{u,f,g,\tau})$, it follows from (4.37) that

$$J(\omega_\tau, u \circ \mathbf{T}_\tau^{-1}, f, g) = \tilde{H}(\tau, u, f, g, y^{u,f,g,\omega}, v^{u,f,g,\omega}, p^{u,f,g,\tau}, w^{u,f,g,\tau}).$$

Hence,

$$J_1(\omega_\tau, f, g) = \inf_{u \in U_{ad}} \tilde{H}(\tau, u, f, g, y^{u,f,g,\omega}, v^{u,f,g,\omega}, p^{u,f,g,\tau}, w^{u,f,g,\tau}). \quad (4.38)$$

Choosing $\overline{u} := \overline{u}^{f,g,\tau}$ in (4.38) with

$$(\tau, u, f, g, y^{u,f,g,\omega}, v^{u,f,g,\omega}, p^{u,f,g,\tau}, w^{u,f,g,\tau})$$

replaced by

$$(\tau_n, \overline{u}_n, f_n, g_n, y^{\overline{u}_n,f_n,g_n,\omega}, v^{\overline{u}_n,f_n,g_n,\omega}, p^{\overline{u}_n,f_n,g_n,\tau_n}, w^{\overline{u}_n,f_n,g_n,\tau_n})$$

and substituting in (4.29), we have

$$\begin{aligned}
&H(\tau_n, \overline{u}_n, f_n, g_n) = \int_{\Omega_T} \frac{1}{2}\zeta(\tau_n)(|y^{\overline{u}_n,f_n,g_n,\omega}|^2 + |v^{\overline{u}_n,f_n,g_n,\omega}|^2 + \alpha|\overline{u}_n|^2)\, dxdt \\
&+ \int_{\Omega_T} \zeta(\tau_n)\frac{\partial v^{\overline{u}_n,f_n,g_n,\omega}}{\partial t} p^{\overline{u}_n,f_n,g_n,\tau_n} + A(\tau_n)\nabla y^{\overline{u}_n,f_n,g_n,\omega} \cdot \nabla p^{\overline{u}_n,f_n,g_n,\tau_n}\, dxdt \\
&+ \int_{\Omega_T} \Big(- \zeta(\tau_n)\chi_\omega \overline{u}_n p^{\overline{u}_n,f_n,g_n,\tau_n} + \zeta(\tau_n)\frac{\partial y^{\overline{u}_n,f_n,g_n,\omega}}{\partial t} w^{\overline{u}_n,f_n,g_n,\tau_n} \\
&- \zeta(\tau_n) v^{\overline{u}_n,f_n,g_n,\omega} w^{\overline{u}_n,f_n,g_n,\tau_n} \Big)\, dxdt + \int_\Omega \Big[\zeta(\tau_n)\Big(y^{\overline{u}_n,f_n,g_n,\omega}(x,0) \\
&- f_n \circ \mathbf{T}_{\tau_n}\Big) w^{\overline{u}_n,f_n,g_n,\tau_n}(x,0) + \zeta(\tau_n)\Big(v^{\overline{u}_n,f_n,g_n,\omega}(x,0) - g_n \circ \mathbf{T}_{\tau_n}\Big) p^{\overline{u}_n,f_n,g_n,\tau_n}(x,0)\Big]\, dx. \quad (4.39)
\end{aligned}$$

From (4.5) as $\tau_n \to 0^+$, we have $\zeta(\tau_n) \to 1, A(\tau_n) \to I$. Utilizing this result in (4.39), and re-arranging the terms, we obtain

$$\begin{aligned}
&\frac{H(\tau_n, \overline{u}_n, f_n, g_n) - H(0, \overline{u}_n, f_n, g_n)}{\tau_n} \\
&= \int_{\Omega_T} \frac{\zeta(\tau_n) - 1}{\tau_n} \cdot \frac{1}{2}\Big(|y^{\overline{u}_n,f_n,g_n,\omega}|^2 + |v^{\overline{u}_n,f_n,g_n,\omega}|^2 + \alpha|\overline{u}_n|^2 \Big)\, dxdt \\
&+ \int_{\Omega_T} \frac{\zeta(\tau_n) - 1}{\tau_n} \frac{\partial v^{\overline{u}_n,f_n,g_n,\omega}}{\partial t} p^{\overline{u}_n,f_n,g_n,\tau_n} + \frac{A(\tau_n) - I}{\tau_n} \nabla y^{\overline{u}_n,f_n,g_n,\omega} \cdot \nabla p^{\overline{u}_n,f_n,g_n,\tau_n} \\
&- \frac{\zeta(\tau_n) - 1}{\tau_n} \chi_\omega \overline{u}_n p^{\overline{u}_n,f_n,g_n,\tau_n}\, dxdt + \int_{\Omega_T} \frac{\zeta(\tau_n) - 1}{\tau_n} \Big(\frac{\partial y^{\overline{u}_n,f_n,g_n,\omega}}{\partial t} w^{\overline{u}_n,f_n,g_n,\tau_n} \\
&- v^{\overline{u}_n,f_n,g_n,\omega} w^{\overline{u}_n,f_n,g_n,\tau_n} \Big)\, dxdt + \int_\Omega \frac{\zeta(\tau_n) - 1}{\tau_n} \Big(\Big(y^{\overline{u}_n,f_n,g_n,\omega}(x,0) \\
&- f_n \circ \mathbf{T}_{\tau_n} \Big) w^{\overline{u}_n,f_n,g_n,\tau_n}(x,0) + \Big(v^{\overline{u}_n,f_n,g_n,\omega}(x,0) - g_n \circ \mathbf{T}_{\tau_n} \Big) p^{\overline{u}_n,f_n,g_n,\tau_n}(x,0) \Big)\, dx \\
&- \int_\Omega \Big(\frac{f_n \circ \mathbf{T}_{\tau_n} - f_n}{\tau_n} w^{\overline{u}_n,f_n,g_n,\tau_n}(x,0) + \frac{g_n \circ \mathbf{T}_{\tau_n} - g_n}{\tau_n} p^{\overline{u}_n,f_n,g_n,\tau_n}(x,0) \Big)\, dx. \quad (4.40)
\end{aligned}$$

Note that

$$g_n \circ \mathbf{T}_{\tau_n} = \frac{\partial}{\partial t}\left(y^{u\circ \mathbf{T}_{\tau_n}^{-1}, f_n, g_n, \omega}(x,0) \circ \mathbf{T}_{\tau_n} \right)$$

and

$$\frac{g_n \circ \mathbf{T}_{\tau_n} - g_n}{\tau_n} = \frac{\partial}{\partial t}\left(\frac{y^{u\circ \mathbf{T}_{\tau_n}^{-1}, f_n, g_n, \omega}(x,0) \circ \mathbf{T}_{\tau_n} - y^{u\circ \mathbf{T}_{\tau_n}^{-1}, f_n, g_n, \omega}(x,0)}{\tau_n} \right)$$

since $\mathbf{T}_\tau$ is independent of t. Using these results, Lemmas 4.2 and 4.9, the right-hand side of (4.40) converges to

$$\begin{aligned}
&\int_{\Omega_T} \operatorname{div}(\mathbf{X}) \left(\frac{1}{2}|\overline{y}^{f,g,\omega}|^2 + \frac{1}{2}|\overline{v}^{f,g,\omega}|^2 + \frac{\alpha}{2}|\overline{u}^{f,g,\omega}|^2 + \frac{\partial \overline{v}^{f,g,\omega}}{\partial t}\overline{p}^{f,g,\omega} \right. \\
&\left. + \frac{\partial \overline{y}^{f,g,\omega}}{\partial t}\overline{w}^{f,g,\omega} - \overline{v}^{f,g,\omega}\overline{w}^{f,g,\omega} + \nabla \overline{y}^{f,g,\omega} \cdot \nabla \overline{p}^{f,g,\omega} - \chi_\omega \overline{u}^{f,g,\omega}\overline{p}^{f,g,\omega} \right) dxdt \\
&- \int_{\Omega_T} \left(\partial \mathbf{X} \nabla \overline{y}^{f,g,\omega} \cdot \nabla \overline{p}^{f,g,\omega} + \partial \mathbf{X}^T \nabla \overline{y}^{f,g,\omega} \cdot \nabla \overline{p}^{f,g,\omega} + \frac{1}{T}\nabla f \cdot \mathbf{X}\overline{w}^{f,g,\omega}(x,0) \right. \\
&\left. + \frac{1}{T}\nabla g \cdot \mathbf{X}\overline{p}^{f,g,\omega}(x,0) \right) dxdt. \qquad (4.41)
\end{aligned}$$

Integrating the fourth and fifth terms of (4.41) by partial integration in time t, and using the facts that $\overline{p}^{f,g,\omega}(x,T) = 0, \overline{w}^{f,g,\omega}(x,T) = 0,\ A\colon B = \sum_{i,l=1}^2 a_{il}b_{il}$ and $a \otimes b\colon A = a \cdot Ab,\ a,b \in \mathbb{R}^2,\ A,B \in \mathbb{R}^{2\times 2}$, we have

$$\begin{aligned}
&\int_{\Omega_T} \left(\left(\tfrac{1}{2}|\overline{y}^{f,g,\omega}|^2 + \tfrac{1}{2}|\overline{v}^{f,g,\omega}|^2 + \tfrac{\alpha}{2}|\overline{u}^{f,g,\omega}|^2 - \overline{v}^{f,g,\omega}\tfrac{\partial \overline{p}^{f,g,\omega}}{\partial t} \right.\right. \\
&- \overline{y}^{f,g,\omega}\tfrac{\partial \overline{w}^{f,g,\omega}}{\partial t} + \nabla \overline{y}^{f,g,\omega} \cdot \nabla \overline{p}^{f,g,\omega} - \chi_\omega \overline{u}^{f,g,\omega}\overline{p}^{f,g,\omega} - \overline{v}^{f,g,\omega}\overline{w}^{f,g,\omega} - \tfrac{1}{T} g\overline{p}^{f,g,\omega}(x,0) \\
&\left.\left. - \tfrac{1}{T} f\overline{w}^{f,g,\omega}(x,0) \right) I - \nabla \overline{y}^{f,g,\omega} \otimes \nabla \overline{p}^{f,g,\omega} - \nabla \overline{p}^{f,g,\omega} \otimes \nabla \overline{y}^{f,g,\omega} \right) : \partial \mathbf{X} \\
&- \tfrac{1}{T}\left(\nabla f \overline{w}^{f,g,\omega}(x,0) + \nabla g \overline{p}^{f,g,\omega}(x,0) \right) \cdot \mathbf{X}\ dxdt.
\end{aligned}$$

Thus, we have the tensor representation (4.35)–(4.36). Hence,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{H(\tau_n, \overline{u}_n, f_n, g_n) - H(0, \overline{u}_n, f_n, g_n)}{\tau_n} \\
&= \int_{\Omega_T} S_1\left(\overline{y}^{f,g,\omega}, \overline{v}^{f,g,\omega}, \overline{p}^{f,g,\omega}, \overline{w}^{f,g,\omega}, \overline{u}^{f,g,\omega} \right) : \partial \mathbf{X} + S_0(f,g) \cdot \mathbf{X}\ dxdt. \qquad (4.42)
\end{aligned}$$

Suppose that $\overline{u}_{n,0} := \overline{u}^{f_n,g_n,0}$. Then similarly, modifying $\overline{u}_n$ as $\overline{u}_{n,0}$, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \frac{H(\tau_n, \overline{u}_{n,0}, f_n, g_n) - H(0, \overline{u}_{n,0}, f_n, g_n)}{\tau_n} \\
&= \int_{\Omega_T} S_1\left(\overline{y}^{f,g,\omega}, \overline{v}^{f,g,\omega}, \overline{p}^{f,g,\omega}, \overline{w}^{f,g,\omega}, \overline{u}^{f,g,\omega} \right) : \partial \mathbf{X} + S_0(f,g) \cdot \mathbf{X}\ dxdt. \qquad (4.43)
\end{aligned}$$

Let $\{f_n\}$ and $\{g_n\}$ be constant sequences. Then, it is clearly seen that $H(\tau_n, \overline{u}_n, f_n, g_n) - H(0, \overline{u}_n, f_n, g_n)$ in (4.42) and $H(\tau_n, \overline{u}_{n,0}, f_n, g_n) - H(0, \overline{u}_{n,0}, f_n, g_n)$ in (4.43) are equal. Hence, (H2) is satisfied. Utilizing Lemma 4.7, we obtain LHS of (4.42) and (4.43) as $\partial_\tau H(0^+, \overline{u}^{0,f,g}, f, g)$ and $\partial_\tau H(0^+, \overline{u}^{f,g,0}, f, g)$, respectively. Hence, (H3) is satisfied. □

As a consequence of Theorem 4.11, we obtain the directional derivative of J_1.

Corollary 4.12. *Let the hypotheses of Theorem 4.11 hold. Let* $(f,g) \in H_0^1(\Omega) \times L^2(\Omega) := \mathbb{V}$ *be given. Then the directional derivative of* $J_1(\omega, f, g)$ *at* ω *in the direction* $\mathbf{X} \in \mathring{C}^{0,1}(\overline{\Omega}, \mathbb{R}^2)$ *is given by*

$$DJ_1(\omega, f, g)(\mathbf{X}) = \int_{\Omega_T} S_1\left(\overline{y}^{f,g,\omega}, \overline{v}^{f,g,\omega}, \overline{p}^{f,g,\omega}, \overline{w}^{f,g,\omega}, \overline{u}^{f,g,\omega}\right) : \partial\mathbf{X} + S_0(f,g) \cdot \mathbf{X} dxdt, \tag{4.44}$$

where $S_1\left(\overline{y}^{f,g,\omega}, \overline{v}^{f,g,\omega}, \overline{p}^{f,g,\omega}, \overline{w}^{f,g,\omega}, \overline{u}^{f,g,\omega}\right)$ *and* $S_0(f,g)$ *are defined by (4.36).*

Proof. For a constant $R > 0$, we note that

$$\max_{\substack{f \in K_1, g \in K_2 \\ \|f\|_{H_0^1(\Omega)} \leq R, \|g\|_{L^2(\Omega)} \leq R}} J_1(\omega, f, g) = R^2 \max_{\substack{f \in \frac{1}{R}K_1, g \in \frac{1}{R}K_2 \\ \|f\|_{H_0^1(\Omega)} \leq 1, \|g\|_{L^2(\Omega)} \leq 1}} J_1(\omega, f, g). \tag{4.45}$$

From (4.45) and by the hypotheses of Theorem 4.11, we deduce that $\frac{f}{R} \in K_1$ and $\frac{g}{R} \in K_2$ with $\|\frac{f}{R}\|_{H_0^1(\Omega)} \leq 1$ and $\|\frac{g}{R}\|_{L^2(\Omega)} \leq 1$. Thus, we have the singleton $\{K_1, K_2\} := \{(f,g)\}$. So, for all $\omega \in E(\Omega)$, we have

$$J_2(\omega) = \max_{f \in K_1, g \in K_2} J_1(\omega, f, g) = J_1(\omega, f, g).$$

Hence, we deduce that $\mathfrak{X}_2(\omega) = \{(f,g)\}$. Since $\mathfrak{X}_2(\omega)$ is a singleton, (4.44) follows by Theorem 4.11. $\square$

As a further consequence of Theorem 4.11, we write (4.35) as an integral over $\partial\omega$. To this end, we require that ω and Ω are C^2 domains. Additionally, for any two sets ω and Ω, the notation $\omega \Subset \Omega$ will be used to mean that ω is compactly contained in Ω. In other words, $\omega \Subset \Omega$ if $\overline{\omega} \subset \Omega$ and $\overline{\omega}$ is compact.

Corollary 4.13. *Let* $f \in K_1, g \in K_2$ *and* $\mathbf{X} \in \mathring{C}^{0,1}(\overline{\Omega}, \mathbb{R}^2)$ *be given. Assume that* $\omega \Subset \Omega$ *and* Ω *are* C^2 *domains.*

(a) *Given* $(f,g) \in \mathfrak{X}_2(\omega)$*, define* $\hat{S}_1(f,g)$ *and* $\hat{S}_0(f,g)$ *by* $\hat{S}_1(f,g) := \int_0^T S_1(f,g)(s)\, ds$ *and* $\hat{S}_0(f,g) := \int_0^T S_0(f,g)(s)\, ds$*, respectively. Then we have*

$$\hat{S}_1(f,g)|_\omega \in W^{1,1}(\omega, \mathbb{R}^{2\times 2}), \hat{S}_1(f,g)|_{\Omega\setminus\overline{\omega}} \in W^{1,1}(\Omega \setminus \overline{\omega}, \mathbb{R}^{2\times 2}), \hat{S}_0(f,g)|_\omega \in L^2(\omega, \mathbb{R}^2), \tag{4.46}$$

$$-\mathrm{div}(\hat{S}_1(f,g)) + \hat{S}_0(f,g) = 0 \text{ a.e. in } \omega \cup (\Omega \setminus \overline{\omega}). \tag{4.47}$$

Moreover, (4.35) can be written as

$$DJ_2(\omega)(\mathbf{X}) = \max_{(f,g) \in \mathfrak{X}_2(\omega)} -\int_{\partial\omega} \int_0^T \overline{u}^{f,g,\omega}(t) \overline{p}^{f,g,\omega}(t)(\mathbf{X} \cdot \nu)\, dtds, \tag{4.48}$$

for $\mathbf{X} \in \mathring{C}^{0,1}(\overline{\Omega}, \mathbb{R}^2)$*, with* ν *the outer normal to* ω*. We denote the jump of* $\hat{S}_1(f,g)\nu$ *across* $\partial\omega$ *by* $[\hat{S}_1(f,g)\nu] := \hat{S}_1(f,g)|_\omega \nu - \hat{S}_1(f,g)|_{\Omega\setminus\omega}\nu$.

(b) *We have that (4.44) can be written as*

$$DJ_1(\omega, f, g)(\mathbf{X}) = -\int_{\partial\omega} \int_0^T \overline{u}^{f,g,\omega}(t) \overline{p}^{f,g,\omega}(t)(\mathbf{X} \cdot \nu)\, dtds, \tag{4.49}$$

for $\mathbf{X} \in \mathring{C}^{0,1}(\overline{\Omega}, \mathbb{R}^2)$.

We begin by stating an important lemma, the so-called Nagumo's lemma (see e.g., [15]) before proving Corollary 4.13. The outer normal to $\partial\mathbb{R}^2$ will be denoted by ν.

Lemma 4.14. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class C^k, $k \geq 1$. Suppose that $\mathbf{X} \in \mathring{C}^{0,1}(\overline{\mathbb{R}^2}, \mathbb{R}^2)$ is a vector field satisfying*

$$\mathbf{X}(x) \cdot \nu(x) = 0, \text{ for all } x \in \partial\mathbb{R}^2.$$

Then the flow Φ_τ of $\mathbf{X}$ satisfies

$$\Phi_\tau(\Omega) = \Omega \text{ and } \Phi_\tau(\partial\Omega) = \partial\Omega, \text{ for all } \tau.$$

Proof of Corollary 4.13. We prove (4.46)–(4.48) as follows. By Nagumo's lemma, we have $DJ_2(\omega)(\mathbf{X}) = 0$, for all $\mathbf{X} \in C_c^1(\Omega, \mathbb{R}^2)$. Using this condition and definitions of $\hat{S}_1(f,g)$ and $\hat{S}_0(f,g)$ in (4.35), we see that

$$\int_\Omega \hat{S}_1(f,g) \colon \partial\mathbf{X} + \hat{S}_0(f,g) \cdot \mathbf{X} \; dx = 0, \tag{4.50}$$

for all $\mathbf{X} \in C_c^1(\Omega, \mathbb{R}^2)$. Integrating the first term in (4.50) by partial integration and using $\mathbf{X}|_{\partial\Omega}= 0$, we have

$$\int_\Omega (-\mathrm{div}(\hat{S}_1(f,g)) + \hat{S}_0(f,g)) \cdot \mathbf{X} \; dx = 0, \tag{4.51}$$

for all $\mathbf{X} \in C_c^1(\Omega, \mathbb{R}^2)$. Since $\mathbf{X} \in C_c^1(\Omega, \mathbb{R}^2)$, applying the fundamental lemma of calculus of variations on (4.51) gives (4.47). Further, since $y, p \in H^2(\Omega) \cap H_0^1(\Omega)$ follows from elliptic regularity theory (see e.g., [6, p.317]), we have that (4.46) holds. Thus, noting that $\Omega = \omega \cup (\Omega \setminus \overline{\omega})$ and by partial integration, we have for all $\mathbf{X} \in C_c^1(\Omega, \mathbb{R}^2)$,

$$\begin{aligned} DJ_2(\omega)(\mathbf{X}) &= \max_{(f,g)\in\mathfrak{X}_2(\omega)} \int_\Omega \hat{S}_1(f,g) \colon \partial\mathbf{X} + \hat{S}_0(f,g) \cdot \mathbf{X} \; dx, \\ &= \max_{(f,g)\in\mathfrak{X}_2(\omega)} \Bigg(\int_\omega \Bigg(-\mathrm{div}(\hat{S}_1(f,g)) + \hat{S}_0(f,g) \Bigg) \cdot \mathbf{X} \; dx \\ &+ \int_{\Omega\setminus\overline{\omega}} \Bigg(-\mathrm{div}(\hat{S}_1(f,g)) + \hat{S}_0(f,g) \Bigg) \cdot \mathbf{X} \; dx + \int_{\partial\omega} [\hat{S}_1(f,g)\nu] \cdot \mathbf{X} \; ds \Bigg), \\ &\overset{(4.47)}{=} \max_{(f,g)\in\mathfrak{X}_2(\omega)} \int_{\partial\omega} [\hat{S}_1(f,g)\nu] \cdot \mathbf{X} \; ds. \end{aligned} \tag{4.52}$$

Since (4.46) holds, it follows that

$$\mathbf{T}_\tau(f,g) := \hat{S}_1(f,g) + \int_0^T \chi_\omega \overline{u}^{f,g,\omega}(t)\overline{p}^{f,g,\omega}(t) \; dt \in W^{1,1}(\omega, \mathbb{R}^{2\times 2}). \tag{4.53}$$

So, $\mathbf{T}_\tau(f,g)\nu = 0$ on $\partial\omega$. Hence, it is easy to see from (4.53) that

$$[\hat{S}_1(f,g)\nu] = -\Bigg(\int_0^T \chi_\omega \overline{u}^{f,g,\omega}(t)\overline{p}^{f,g,\omega}(t) \; dt \Bigg)\nu. \tag{4.54}$$

Since $\mathbf{X}$ and ν are independent of time t, substituting (4.54) in (4.52) gives

$$DJ_2(\omega)(\mathbf{X}) = \max_{(f,g)\in\mathfrak{X}_2(\omega)} - \int_{\partial\omega} \int_0^T \overline{u}^{f,g,\omega}(t)\overline{p}^{f,g,\omega}(t)(\mathbf{X}\cdot\nu) \; dtds,$$

as was to be proved.

The proof of (4.49) is similar to the proof of Corollary 4.12. □

4.5. **Gradient Algorithm for Optimal Actuator Placement.** Here, we present the steps of a gradient-based algorithm for optimal actuator placement. The version of the algorithm is summarized in Algorithm 1. It is important to note that we can also use J_2 in this algorithm to investigate the optimal actuator placement by replacing J_1 with J_2.

Algorithm 1 Shape derivative-based gradient algorithm for optimal actuator placement

Require: $\omega_0 \in E(\Omega), f, g$, tolerance $\varepsilon > 0, k = 0, \lambda, d_0 := -\nabla J_1(\omega_0, f, g)$.
  while $|d_k| \geq \varepsilon$ **do**
    if $J_1((\mathrm{id} + \lambda d_k)(\omega_k), f, g) < J_1(\omega_k, f, g)$ **then**
      $d_k = -\nabla J_1(\omega_k, f, g)$
      $\omega_{k+1} = (\mathrm{id} + \lambda d_k)(\omega_k)$
      $k := k + 1$
    end if
  end while
  return optimal actuator placement ω_{k+1}

5. Numerical Examples

5.1. **Discretization.** Let step sizes be h in space and $\triangle t$ in time, i.e., $\triangle x_1 = \triangle x_2 = h$ and $t_k = k\triangle t$. Then, discretizing (3.6) and (3.7) using finite differences, we have for $k = 1, 2, \ldots, M-1$

$$\begin{aligned} \mathbf{y}_h^{k+1} &= \mathbf{y}_h^k + \triangle t \mathbf{v}_h^k, \\ \mathbf{v}_h^{k+1} &= \mathbf{v}_h^k + A_r \mathbf{y}_h^k + \triangle t \chi_\omega \mathbf{u}_h^k, \\ \mathbf{y}_h^1 &= \mathbf{f}_h, \\ \mathbf{v}_h^1 &= \mathbf{g}_h, \end{aligned} \tag{5.1}$$

and for $k = M, M-1, \ldots, 2$

$$\begin{aligned} \mathbf{p}_h^{k-1} &= \mathbf{p}_h^k + \triangle t(\mathbf{w}_h^k - \mathbf{v}_h^k), \\ \mathbf{w}_h^{k-1} &= \mathbf{w}_h^k + A_r \mathbf{p}_h^k - \triangle t \mathbf{y}_h^k, \\ \mathbf{p}_h^M &= \mathbf{0}, \\ \mathbf{w}_h^M &= \mathbf{0}, \end{aligned} \tag{5.2}$$

respectively, where

$$\begin{aligned} \mathbf{y}_h &= (y_{11}, y_{12}, \ldots, y_{(N-1)^2(N-1)^2})^\top, \mathbf{v}_h = (v_{11}, v_{12}, \ldots, v_{(N-1)^2(N-1)^2})^\top, \\ \mathbf{u}_h &= (u_{11}, u_{12}, \ldots, u_{(N-1)^2(N-1)^2})^\top, \mathbf{f}_h = (f_{11}, f_{12}, \ldots, f_{(N-1)^2(N-1)^2})^\top, \\ \mathbf{g}_h &= (g_{11}, g_{12}, \ldots, g_{(N-1)^2(N-1)^2})^\top, \mathbf{p}_h = (p_{11}, p_{12}, \ldots, p_{(N-1)^2(N-1)^2})^\top, \\ \mathbf{w}_h &= (w_{11}, w_{12}, \ldots, w_{(N-1)^2(N-1)^2})^\top, r = \frac{\triangle t}{h^2} \end{aligned}$$

and

$$A_r = \begin{pmatrix} \mathbf{B} & \mathbf{I} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{I} & \mathbf{B} & \mathbf{I} & \ddots & \vdots \\ \mathbf{0} & \mathbf{I} & \ddots & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \mathbf{B} & \mathbf{I} \\ \mathbf{0} & \ldots & \mathbf{0} & \mathbf{I} & \mathbf{B} \end{pmatrix} \quad \text{with } \mathbf{B} = \begin{pmatrix} -4 & 1 & 0 & \ldots & 0 \\ 1 & -4 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -4 & 1 \\ 0 & \ldots & 0 & 1 & -4 \end{pmatrix}$$

and $\mathbf{I}$ is the identity matrix. The matrix A_r is of size $(N-1)^2 \times (N-1)^2$ while matrices $\mathbf{B}$, $\mathbf{I}$ and $\mathbf{0}$ are of size $(N-1) \times (N-1)$. The discrete functionals of J_1 and J_2 are

$$J_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h) = \frac{1}{2} \min_{\mathbf{u}_h \in U_{ad}} \int_0^T \mathbf{y}_h(t)^\top \mathbf{y}_h(t) + \mathbf{v}_h(t)^\top \mathbf{v}_h(t) + \alpha \chi_\omega \mathbf{u}_h(t)^\top \chi_\omega \mathbf{u}_h(t) dt \tag{5.3}$$

and

$$J_{2,h}(\omega) = \max_{\mathbf{f}_h, \mathbf{g}_h} J_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h), \tag{5.4}$$

respectively. The discrete derivatives $J_{1,h}$ and $J_{2,h}$ are given by

$$DJ_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h)(\mathbf{X}) = -\int_{\partial\omega}\int_0^T \mathbf{u}_h(s,t)^\top \mathbf{p}_h(s,t)(\mathbf{X}\cdot\nu)\ dtds \tag{5.5}$$

and

$$DJ_{2,h}(\omega)(\mathbf{X}) = \max_{\mathbf{f}_h, \mathbf{g}_h} J_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h),$$

for $\mathbf{X} \in \mathring{C}^{0,1}(\overline{\Omega}, \mathbb{R}^2)$, respectively. The vector $b \in \mathbb{R}^2$ has components

$$b_j := -\int_{\partial\omega}\int_0^T \mathbf{u}_h(s,t)^\top \mathbf{p}_h(s,t)(e_j\cdot\nu)\ dtds, j = 1, 2,$$

where e_j is the jth element of the standard basis of $\mathbb{R}^2$.

5.2. **Examples.** We illustrate the actuator placement optimizations for two cases of initial conditions f and g. In all the experiments, the actuators ω_1, and ω_2 each of fixed size 0.2×0.2 are placed on the domain and moved along the descent direction $x_1 = x_2$. We consider two actuators without overlap such that they move into their optimal locations. We set the tolerance ε to 10^{-4} and N to 8.

Example 5.21. We consider the case

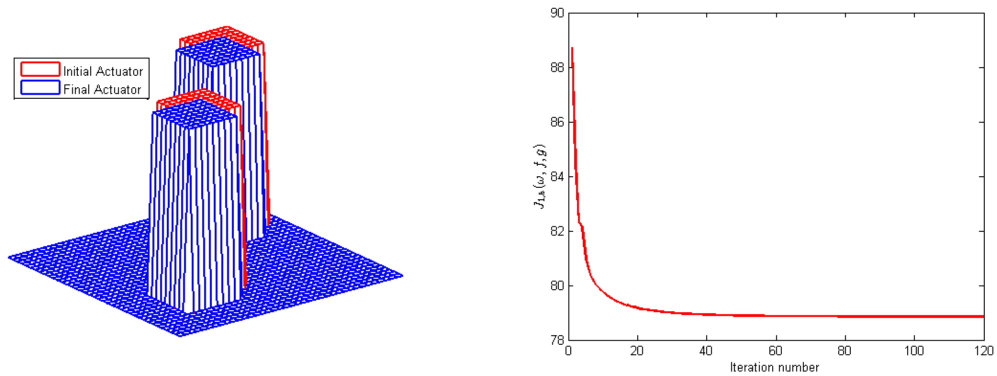
$$\begin{aligned} y(x_1, x_2, 0) &= \sin \pi x_1 \sin \pi x_2, \qquad 0 \le x_1, x_2 \le 1,\\ v(x_1, x_2, 0) &= \frac{\pi c}{20}\sin \pi x_1 \sin \pi x_2, \quad 0 \le x_1, x_2 \le 1, \end{aligned}$$

so that the initial speed $v(x_1, x_2, 0)$ varies with the speed of wave $1 \le c \le \frac{20}{\pi}$. First, we start by investigating the optimal actuator placement using $J_{1,h}$. For initial actuators ω_1, ω_2 centered at $(0.4, 0.4)$ and $(0.825, 0.825)$, respectively, a shape optimization Algorithm 1 is utilized. The results are presented in Figure 2. It is observed from Figure 2 that as the actuators move toward the optimal locations in the subsequent iterations (see Figure 2(a)), the functional $J_{1,h}$ decays until a stationary point is reached, see Figure 2(b). The optimization algorithm converges after 120 iterations. The optimal actuators are centered at $(0.325, 0.325)$ and $(0.75, 0.75)$, respectively.

Next, we perform numerical experiments using $J_{2,h}$ but with initial actuators ω_1, ω_2 centered at $(0.2, 0.2)$ and $(0.825, 0.825)$, respectively. Algorithm 1 is run until the set criterion is achieved. The results are depicted in Figure 3. From this figure, we see that as the actuators move toward the optimal locations, see Figure 3(a), the functional $J_{2,h}$ decays until a stationary point is reached, see Figure 3(b). The convergence of the optimization algorithm occurs after 73 iterations. The final actuator locations are found at $(0.325, 0.325)$ and $(0.75, 0.75)$, respectively. This is consistent with the result obtained by using $J_{1,h}$.

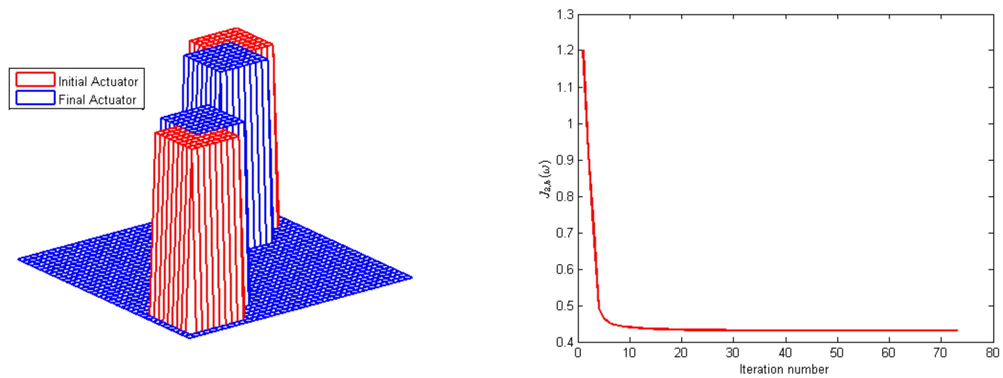
Lastly, the results of the experiments to investigate the influence of the wave speed are shown in Figure 4 and Table 1.

From Table 1, we see that when ω_1 is placed at $(0.4, 0.4)$, and ω_2 at $(0.825, 0.825)$ (see Figure 4(a)), the least values of both $J_{1,h}$ and $J_{2,h}$ are obtained. Furthermore, it is observed that $J_{1,h}$ increases with an increase in the wave speed c, see Figure 4 and Table 1. We also note from Table 1 that the least values of $J_{2,h}(\omega)$ after 120 iterations are the same. This confirms the fact that the dependence of $J_{1,h}$ on the initial conditions is averaged out.



(a) (b)

FIGURE 2. (a) The initial actuator center locations: $(0.4, 0.4)$, $(0.825, 0.825)$ (red) and final actuator center locations: $(0.325, 0.325)$, $(0.75, 0.75)$ (blue). (b) The history of cost functional $J_{1,h}$, as the actuators move from the initial to the optimal actuator locations. The speed of the wave is set to $c = 1$.



(a) (b)

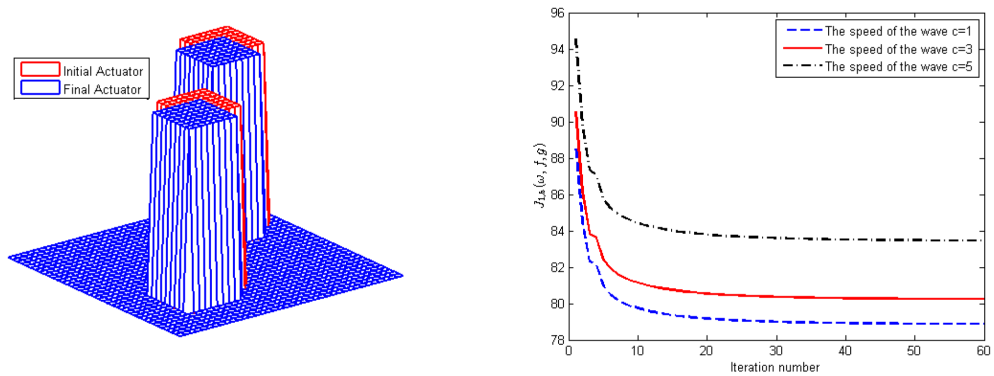
FIGURE 3. (a) The initial actuator center locations: $(0.2, 0.2)$, $(0.825, 0.825)$ (red) and final actuator center locations: $(0.325, 0.325)$, $(0.75, 0.75)$ (blue). (b) The history of cost functional $J_{2,h}$, as the actuators move from the initial to the optimal actuator locations. The speed of the wave is set to $c = 1$.

Example 5.22. In this example, we set

$$\begin{aligned} y(x_1, x_2, 0) &= x_1 x_2 (1 - x_1)(1 - x_2), && 0 \leq x_1, x_2 \leq 1, \\ v(x_1, x_2, 0) &= \frac{1}{2} \sin(x_1(1 - x_1) x_2 (1 - x_2)), && 0 \leq x_1, x_2 \leq 1, \end{aligned}$$

so that the initial conditions of the dynamics satisfy Dirichlet boundary conditions. Therefore, the optimal actuator center locations are expected at points different from the boundary of the domain. First, we start by investigating the optimal actuator placement using $J_{1,h}$. With initial actuator center locations ω_1, ω_2 at $(0.2, 0.2)$ and $(0.825, 0.825)$, respectively, the results are shown in Figure 5.

The minimum value of $J_{1,h}$ occurs when the actuators are placed at $(0.3, 0.3)$ and $(0.7, 0.7)$.



(a) (b)

FIGURE 4. (a) The initial actuator center locations: $(0.4, 0.4)$, $(0.825, 0.825)$ (red) and final actuator center locations: $(0.325, 0.325)$, $(0.75, 0.75)$ (blue). (b) Demonstration of the influence of wave speed on the history of cost functional $J_{1,h}$, as the actuators move from the initial to the final actuator locations.

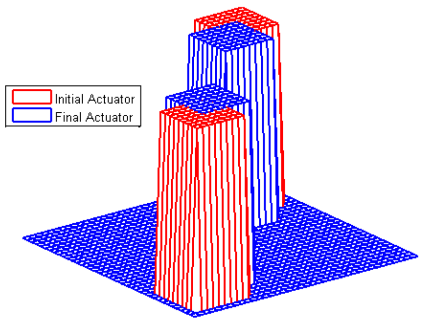
TABLE 1. The minimum values of functionals $J_{1,h}$ and $J_{2,h}$ after 120 iterations for the given speed of wave c. The measure of the cost of control is set to $\alpha = 10^{-4}$.

c	$J_{1,h}(\omega, f, g)$	$J_{2,h}(\omega)$
1	78.8529	0.4307
3	80.2176	0.4307
5	83.4128	0.4307

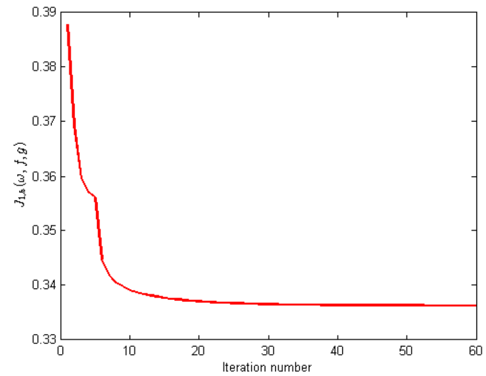
Next, we perform a numerical experiment using $J_{2,h}$. With initial actuator center locations at $(0.2, 0.2)$ and $(0.825, 0.825)$, we run Algorithm 1 until the set criterion is achieved. The results are given in Figure 6. From this figure, we see that the functional $J_{2,h}$ decays until a stationary point is reached. The convergence of the optimization algorithm occurs after 50 iterations. The final actuator locations are found at $(0.3, 0.3)$ and $(0.7, 0.7)$, see Figure 6(a). This is in agreement with the result obtained by using $J_{1,h}$.

6. CONCLUSION

In this paper, we proved important results for the differentiability of functionals J_1 and J_2. The shape derivative is derived using the averaged adjoint approach. We also developed a shape derivative-based-gradient algorithm for determining the optimal actuator placement for the control of vibrations induced by pedestrian-bridge interactions. The algorithm is constructed by embedding the shape sensitivities in a gradient-based method. The numerical results presented illustrate the potential of the shape sensitivities for solving the optimal actuator placement problem whenever the actuator's width is fixed in advance. The optimal actuator design for the wave equation is under our next research study plan. The term "design" here means picking the best domain ω by parametrizing the set of admissible domains.

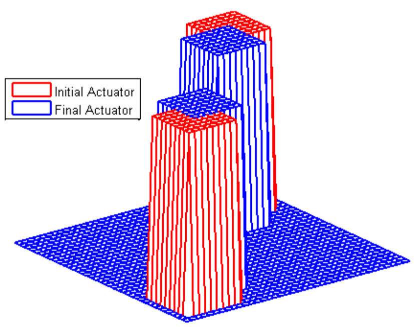


(a)

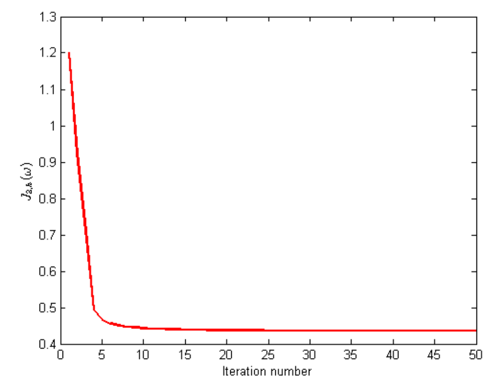


(b)

FIGURE 5. (a) The initial actuator center locations: $(0.2, 0.2)$, $(0.825, 0.825)$ (red) and final actuator center locations: $(0.3, 0.3)$, $(0.7, 0.7)$ (blue). (b) Demonstration of the history of cost functional $J_{1,h}$, as the actuators move from the initial to the optimal actuator locations.



(a)



(b)

FIGURE 6. (a) The initial actuator center locations: $(0.2, 0.2)$, $(0.825, 0.825)$ (red) and final actuator center locations: $(0.3, 0.3)$, $(0.7, 0.7)$ (blue). (b) History of cost functional $J_{2,h}$, as the actuators move from the initial to the optimal actuator locations.

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M. D. Arop, corresponding author, Department of Mathematics, Makerere University, Kampala, Uganda
Current address: Department of Mathematics, Muni University, Arua, Uganda
Email address: d.arop@muni.ac.ug

H. Kasumba, Department of Mathematics, Makerere University, Kampala, Uganda
Email address: henry.kasumba@mak.ac.ug

J. Kasozi, Department of Mathematics, Makerere University, Kampala, Uganda
Email address: juma.kasozi@mak.ac.ug

F. Berntsson, Department of Mathematics, Linköping University, Linköping, Sweden
Email address: fredrik.berntsson@liu.se