Mathematics in Applied Sciences and Engineering
Online First, pp.**1-20**

https://ojs.lib.uwo.ca/mase
https://doi.org/10.5206/mase/16958

# OPTIMAL ACTUATOR DESIGN FOR CONTROL OF VIBRATIONS INDUCED BY PEDESTRIAN-BRIDGE INTERACTIONS

MARTIN DEOSBORNS AROP, HENRY KASUMBA, JUMA KASOZI, AND FREDRIK BERNTSSON

Abstract. In this paper, we are interested in finding an optimal control support design for controlling vibrations due to pedestrian-bridge interactions. Therefore, we derive the topological derivatives of the proposed functionals using the averaged adjoint approach. A numerical algorithm initialized by these sensitivities is used as a solution strategy. The algorithm is tested numerically for two different cases of initial conditions.

## 1. Introduction

An optimal actuator design problem involves finding both the optimal location and shape of the subdomain [25]. This problem arises in many areas of application in science and engineering, for example, in seismic inversion, medical applications, and control and stabilization of waves [23].

Optimal actuator placement problems have been studied extensively as manifested in the engineering literature (see, e.g., [10],[24]) and optimal actuator placement theory of linear distributed systems (see e.g., [15], [16]). On the other hand, optimal actuator design problems have received a relatively low but growing amount of attention. We briefly mention some literature related to optimal actuator design problems as follows.

In a pioneering work, Hébrard and Henrott [11] investigated an optimal shape and position for the stabilization of a string. They considered a one-dimensional wave equation with objective functional defined as an integral of quadratic normalized eigenfunctions. Both the damping term and normalized eigenfunctions were parametrized using characteristic functions. A genetic algorithm was used as a solution strategy for the problem. Numerical results showed that the optimal way of damping the string was to split actuators into many parts.

In related work, Hébrard and Henrott [12] studied optimal shape and position for the stabilization of a string described by the same system given in [11] but with a criterion approximated by a finite sequence of eigenfunctions. Thus, the authors were able to prove the existence and uniqueness of the optimal subdomain. The spillover was that the optimal position for any given mode was the worst for the succeeding mode.

Also, Münch [17] studied the problem of finding the optimal design of the control support for the one-dimensional wave equation as an exact controllability problem. The Hilbert uniqueness method (HUM) was used to reduce the exact controllability problem to an optimal control problem. The shape and topological derivatives of the criterion with respect to the control support were computed based on Céa's method [4]. For numerical realization, a mixture of conjugate gradient algorithms and finite

Received by the editors 20 October 2023; accepted 14 May 2024; published online 24 May 2024.

2020 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.

*Key words and phrases.* Averaged adjoint approach, finite element method, optimal actuator design, shape optimization, wave equation.

M.D. Arop was supported in part by SIDA bilateral programme (2015–2022) with Makerere University; Project 316.

difference schemes were utilized. The optimal control support was obtained for the one-dimensional wave equation.

Relatedly, Münch [18] numerically investigated an optimal actuator design problem for a two-dimensional wave equation with Dirichlet boundary conditions. By initializing a descent algorithm using topological derivatives, the author determined optimal shapes corresponding to the exact controllability problem. The optimal shapes were depending on the initial conditions and final time. The optimal actuators were determined using techniques from optimal control and spectral theory.

In [20], the authors investigated an optimal design of controllers for the one-dimensional wave equation with the aim of minimizing the norm of the control such that the solution is driven exactly to zero at the final time. The HUM was used to reduce the exact controllability problem to an optimal control problem. In addition to minimizing the norm of the control, the authors also investigated the problem of minimizing the supremum of this norm. In their work, frequential analysis approach based on Fourier series expansions was used to solve the optimal control problem while minimizing both the norm of the control and the supremum of this norm. Using this solution procedure, the existence and uniqueness of the optimal subdomain were proved. The latter problem did not admit any optimal solution except at the midpoint of the domain.

Also, an optimal actuator design and placement problem for a linear heat equation was solved using a shape and topology optimization approach [13]. The authors parametrized the actuators by considering controls over some subsets of the domains using characteristic functions.

Similarly, in [19], an optimal actuator design problem governed by a linear parabolic system is discussed. The authors used the moment method to transform the problem to a spectral optimal actuator design which consists of maximizing a criterion over a random initial data. Internal controls were considered using a characteristic function. The existence and uniqueness of the optimal actuators were proved.

Edalatzadeh and Morris [7] considered an optimal actuator design problem with applications in a nonlinear railway track model and semilinear wave models. It was shown that an optimal solution exists under certain assumptions on the nonlinear part and cost function. Using optimal control techniques, the first-order conditions were derived.

An optimal actuator design problem for the control of vibrations governed by the Euler-Bernoulli equation was considered in [8]. The authors used the shape calculus technique to derive a topological derivative and a level-set method for numerical realization. Optimal actuator shapes were obtained provided Kelvin-Voigt damping is taken into consideration.

For this paper, we study a novel problem of optimal actuator design for the control of vibrations induced by pedestrian-bridge interactions. We determine the optimal actuator shape for a linear wave equation using the averaged adjoint approach while numerical realization of the problem is achieved by using a weighted finite difference, and finite element methods [5].

The rest of the paper is organized as follows. In the next section, we introduce the notations that will be relevant for subsequent developments in the sequel and formulate the state and optimization problems. In Section 3, we derive the topological derivatives of the optimization problems. Numerical results that support the theoretical results are given in Section 4. Section 5 comprises conclusions of the work and some remarks for future works.

## 2. Formulation of the Problem

2.1. **Notations.** Let $\mathcal{G}$ be either the domain $\Omega$ or its boundary $\partial\Omega$. Then, we define $L^2(\mathcal{G})$ as a linear space of all measurable functions $y : \mathcal{G} \to \mathbb{R}$ such that

$$\|y\|_{L^2(\mathcal{G})} := \left( \int_{\mathcal{G}} |y|^2 \, dx \right)^{\frac{1}{2}} < \infty.$$

The standard Sobolev space of order $m \in \mathbb{R}^+ \cup \{0\}$, denoted by $H^m(\mathcal{G})$, is defined as

$$H^m(\mathcal{G}) := \{y \in L^2(\mathcal{G}) | D^\gamma y \in L^2(\mathcal{G}), \text{ for all } 0 \le |\gamma| \le m\},$$

where $D^\gamma$ is the weak partial derivative and $\gamma$ is a multi-index. We define the subspace $H_0^1(\mathcal{G})$ of $H^1(\mathcal{G})$ by $H_0^1(\mathcal{G}) := \{y \in H^1(\mathcal{G}) | y = 0 \text{ on } \partial\Omega\}$. The norm $\|\cdot\|_{H^m(\mathcal{G})}$ associated with $H^m(\mathcal{G})$ is given by

$$\|y\|_{H^m(\mathcal{G})} := \sqrt{\sum_{|\gamma| \le m} \int_{\mathcal{G}} |D^\gamma y|^2 \, dx}.$$

Note that $H^0(\mathcal{G}) = L^2(\mathcal{G})$; thus, $\|y\|_{H^0(\mathcal{G})} = \|y\|_{L^2(\mathcal{G})}$. For a functional space $X$, we denote by $L^p(0,T;X)$ $(1 \le p < \infty)$ the space of measurable functions $y : [0,T] \to X$ such that

$$\|y\|_{L^p(0,T;X)} := \left( \int_0^T \|y(\cdot,t)\|_X^p \, dt \right)^{\frac{1}{p}} < \infty,$$

where $T$ is the final time. The space of essentially bounded functions from $[0,T]$ into $X$ is denoted by $L^\infty(0,T;X)$ and is equipped with the norm $\operatorname*{ess\,sup}_{t\in[0,T]} \|y(\cdot,t)\|_X$, where ess sup denotes the essential supremum. We denote the control space by $U := L^2(0,T;L^2(\Omega))$ and the collection of measurable subdomains of $\Omega$ by $E(\Omega)$. We shall use $L^2(L^2(\Omega))$, $L^2(H_0^1(\Omega))$ and $L^\infty(H_0^1(\Omega))$ as the short forms for $L^2(0,T;L^2(\Omega)), L^2(0,T;H_0^1(\Omega))$ and $L^\infty(0,T;H_0^1(\Omega))$, respectively. We denote by $B_\epsilon(\eta_0)$ and $\overline{B}_\epsilon(\eta_0)$ the open and closed balls centered at $\eta_0$ with radius $\epsilon > 0$, respectively. Let $\omega \in E(\Omega)$. Then, set $\omega_\epsilon := \omega \setminus \overline{B}_\epsilon(\eta_0)$ if $\eta_0 \in \omega$ and $\omega_\epsilon := \omega \cup B_\epsilon(\eta_0)$ if $\eta_0 \in \Omega \setminus \overline{\omega}$.

2.2. **Setup of the Problem.** From [3], we consider the following wave equation:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \triangle y = \chi_\omega u, & (x,t) \in \Omega \times (0,T], \\ y = 0, & (x,t) \in \partial\Omega \times (0,T], \\ y(x,0) = f(x), \ \dfrac{\partial y}{\partial t}(x,0) = g(x), \ x \in \Omega, \end{cases} \tag{2.1}$$

where $y = y(x,t)$ denotes the vibrations at position $x$ and time $t$, $u = u(x,t)$ the control variable, $\chi_\omega$ the characteristic function for the domain $\omega \subset \Omega$, and $x \in \mathbb{R}^d, d = 1, 2$.

Since the vibrations may depend on $f, g, u$, and $\omega$, the cost functional to be minimized is given by

$$J(\omega,u,f,g) := \int_0^T \frac{1}{2}\|y^{u,f,g,\omega}(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\frac{dy^{u,f,g,\omega}}{dt}(\cdot,t)\right\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\chi_\omega u(\cdot,t)\|^2_{L^2(\Omega)} dt, \tag{2.2}$$

where $\alpha > 0$ is a given parameter. In particular, let $\omega, f$ and $g$ be fixed. We want to minimize the vibrations and speed while also keeping the cost of control minimum. Thus, by

(i) taking the infimum of the cost $J$ over all controls $u \in U_{ad}$, we obtain the functional $J_1 : E(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ defined by

$$J_1(\omega, f, g) := \inf_{u \in U_{\mathrm{ad}}} J(\omega,u,f,g), \tag{2.3}$$

(ii) choosing the best scenario in (2.3), we introduce a functional $J_2 : E(\Omega) \to \mathbb{R}$ defined by

$$J_2(\omega) := \sup_{f\in K_1, g\in K_2} J_1(\omega, f, g). \tag{2.4}$$

The admissible set of controls $U_{\text{ad}}$ consists of a closed and convex subset of $U$. The weakly compact subsets $K_1$ and $K_2$ of $H_0^1(\Omega)$ and $L^2(\Omega)$ are defined by $K_1 = \{f : \|f\|_{H_0^1(\Omega))} \le 1\}$ and $K_2 = \{g : \|g\|_{L^2(\Omega))} \le 1\}$, respectively.

We now define the optimal actuator design problems for any $\omega \subset \Omega$.

**Definition 2.1.** The optimal actuator design problems related to $J_1$ and $J_2$ are defined by the minimization problems:

$$\inf_{\substack{\omega\in E(\Omega)\\ |\omega|=|\omega_d|}} J_1(\omega, f, g) \tag{2.5}$$

and

$$\inf_{\substack{\omega\in E(\Omega)\\ |\omega|=|\omega_d|}} J_2(\omega), \tag{2.6}$$

where $\omega_d$ denotes the desired actuator with actuator size $|\omega_d| \in (0, |\Omega|)$, respectively.

To determine the optimal actuator design, we derive the topological derivatives of the functionals in the following section.

## 3. Topological Derivatives of the Functionals

It is well known that (2.1) can be reformulated (see, [3]) as the following first-order system:

$$\begin{cases} \dfrac{\partial y^{u,f,g,\omega}}{\partial t} - v^{u,f,g,\omega} = 0, & (x,t) \in \Omega \times (0,T], \\ \dfrac{\partial v^{u,f,g,\omega}}{\partial t} - \triangle y^{u,f,g,\omega} - \chi_\omega u = 0, & (x,t) \in \Omega \times (0,T], \\ y^{u,f,g,\omega}(x,0) = f(x),\ v^{u,f,g,\omega}(x,0) = g(x),\ x \in \Omega, & \\ y^{u,f,g,\omega} = 0, & (x,t) \in \partial\Omega \times (0,T]. \end{cases} \tag{3.1}$$

This reformulation is the basis of the discretization of the optimization problems. From standard techniques (see e.g., [14, pp. 114–115, Thm. 2.1], we obtain the adjoint equations:

$$\begin{cases} \dfrac{\partial p^{\overline{u},f,g,\omega}}{\partial t} - w^{\overline{u},f,g,\omega} = -v^{\overline{u},f,g,\omega}, & (x,t) \in \Omega \times (0,T], \\ \dfrac{\partial w^{\overline{u},f,g,\omega}}{\partial t} - \triangle p^{\overline{u},f,g,\omega} = -y^{\overline{u},f,g,\omega}, & (x,t) \in \Omega \times (0,T], \\ p^{\overline{u},f,g,\omega}(x,T) = 0, \quad w^{\overline{u},f,g,\omega}(x,T) = 0, \quad x \in \Omega, & \\ p^{\overline{u},f,g,\omega} = 0, \qquad (x,t) \in \partial\Omega \times (0,T], & \end{cases} \tag{3.2}$$

and optimality condition

$$\alpha\chi_\omega \overline{u} - \chi_\omega p^{\overline{u},f,g,\omega} = 0, \qquad (x,t) \in \Omega \times (0,T], \tag{3.3}$$

where $p^{\overline{u},f,g,\omega} \in L^2(H_0^1(\Omega))$, $w^{\overline{u},f,g,\omega} \in L^2(L^2(\Omega))$ and $(y^{\overline{u},f,g,\omega}, v^{\overline{u},f,g,\omega}, \overline{u}, p^{\overline{u},f,g,\omega}, w^{\overline{u},f,g,\omega})$ solves (3.1)–(3.3), respectively. The optimality system (3.1)–(3.3) will be utilized to obtain the solution $(y^{\overline{u},f,g,\omega}, v^{\overline{u},f,g,\omega}, \overline{u}, p^{\overline{u},f,g,\omega}, w^{\overline{u},f,g,\omega})$.

**Definition 3.1.** The topological derivative of a shape functional $J : E(\Omega) \to \mathbb{R}$ at $\omega \in E(\Omega)$ in the point $\eta_0 \in \Omega \setminus \partial\omega$ is given by $\mathfrak{T}J(\omega)(\eta_0)$ provided there exists the following limit

$$\mathfrak{T}J(\omega)(\eta_0) := \begin{cases} \lim\limits_{\epsilon \searrow 0} \dfrac{J(\omega \setminus \overline{B}_\epsilon(\eta_0)) - J(\omega)}{|B_\epsilon(\eta_0)|} & \text{if } \eta_0 \in \omega, \\ \lim\limits_{\epsilon \searrow 0} \dfrac{J(\omega \cup B_\epsilon(\eta_0)) - J(\omega)}{|B_\epsilon(\eta_0)|} & \text{if } \eta_0 \in \Omega \setminus \overline{\omega}. \end{cases} \tag{3.4}$$

From Definition 3.1 and the lemmas formulated, we derive the directional derivative of $J_2$ and hence, $J_1$.

The following assumption will be utilized to derive the topological derivative of $J_2$. Note that the minimizer of $\min\limits_{u \in U_{ad}} J(\omega, u, f, g)$ depends also on the shape parameter $\epsilon$. We stress this dependence with the notation $\overline{u}^{f,g,\omega_\epsilon}$, where $\omega = \omega_\epsilon$.

3.1. **Assumption.** Let $\delta > 0$ be so small that $\overline{B}_\delta(\eta_0)$ is compactly contained in $\Omega$, i.e., $\overline{B}_\delta(\eta_0) \Subset \Omega$. Then, we assume that for all $(f, g) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\omega \in E(\Omega)$, we have $\overline{u}^{f,g,\omega} \in C(\overline{B}_\delta(\eta_0))$. Also, we assume that for every sequence $\{\omega_n\}$ in $E(\Omega)$ converging to $\omega \in E(\Omega)$ in characteristics, $f_n \rightharpoonup f$ in $H_0^1(\Omega)$ and $g_n \rightharpoonup g$ in $L^2(\Omega)$, we have

$$\lim_{n\to\infty} ||\overline{u}^{f_n,g_n,\omega_n} - \overline{u}^{f,g,\omega}||_{L^1(0,T;C(\overline{B}_\delta(\eta_0)))} = 0. \tag{3.5}$$

The following lemma will be used to compute the topological derivative.

**Lemma 3.1.** *Let $\delta > 0$ be such that $\overline{B}_\delta(\eta_0) \Subset \Omega$. Then for all $\epsilon_n \in (0, 1]$, $u_n, u \in U$, $f_n, f \in K_1$ and $g_n, g \in K_2$, such that*

$$u_n \rightharpoonup u \text{ in } U, f_n \rightharpoonup f \text{ in } H_0^1(\Omega), g_n \rightharpoonup g \text{ in } L^2(\Omega),\ \epsilon_n \to 0 \text{ as } n \to \infty,$$

*we have*

$$\begin{aligned} &p^{u_n,f_n,g_n,\epsilon_n} \to p^{u,f,g,\omega} \text{ in } L^2(H_0^1(\Omega)) \text{ as } n \to \infty, \\ &p^{u_n,f_n,g_n,\epsilon_n} \rightharpoonup p^{u,f,g,\omega} \text{ in } H^1(0,T;L^2(\Omega)) \text{ as } n \to \infty. \end{aligned} \tag{3.6}$$

*Additionally, there is a subsequence $\{p^{u_{n_k},f_{n_k},g_{n_k},\epsilon_{n_k}}\}$ such that*

$$p^{u_{n_k},f_{n_k},g_{n_k},\epsilon_{n_k}} \to p^{u,f,g,\omega} \text{ in } C([0,T] \times \overline{B}_\delta(\eta_0)) \text{ as } k \to \infty. \tag{3.7}$$

*Proof.* The proof of (3.6) follows from the argument that $p^{u_n,f_n,g_n,\epsilon_n}$ in $L^2(H_0^1(\Omega))$ is bounded. We prove (3.7) as follows. Using the estimate (see [9, pp.391-393, Thm. 6])

$$\|p^{u,f,g,\omega}\|_{L^2(H_0^1(\Omega))} + \left\|\frac{\partial p^{u,f,g,\omega}}{\partial t}\right\|_{L^2(L^2(\Omega))} \le c\|v^{u,f,g,\epsilon} + v^{u,f,g,\omega}\|_{L^2(L^2(\Omega))},$$

we see that $p^{u,f,g,\epsilon}$ is bounded. Note that $p^{u,f,g,\epsilon} \in L^2(H_0^1(\Omega)) \cap H^1(0,T;H_0^1(\Omega)) \cap H^2(0,T;L^2(\Omega))$, see e.g., [9, p.317],[3]. Hence, the orders of differentiability and integrability are $m = 2$ and $p = 2$, respectively. Since for $\Omega \subset \mathbb{R}^d$, $mp = 4 > d$, it follows that $L^2(H_0^1(\Omega)) \cap H^1(0,T;H_0^1(\Omega)) \cap H^2(0,T;L^2(\Omega))$ embeds compactly into $C([0,T] \times B_\delta(\eta_0))$ (see e.g., [22, Thm. 7.1]). By utilizing Rellich-Kondrachov theorem, we conclude that $p^{u_{n_k},f_{n_k},g_{n_k},\epsilon_{n_k}} \to p^{u,f,g,\omega}$ in $C([0,T] \times \overline{B}_\delta(\eta_0))$ as $k \to \infty$. $\square$

Now, we introduce the averaged adjoint equations and Lagrangian in the following subsection.

3.2. **Averaged Adjoint Equations.** To define the averaged adjoint equations, we first, formulate a Lagrangian functional as follows. Consider a fixed open set $\omega \in E(\Omega)$ so that for any point $\eta_0 \in \omega$ or $\eta_0 \in \Omega \setminus \overline{\omega}$, we can find a ball that lies fully in $\omega$ or $\Omega \setminus \overline{\omega}$, respectively.

**Definition 3.2.** Define the parametrized Lagrangian

$$\tilde{H} : [0, \lambda] \times U \times K_1 \times K_2 \times H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$$

by

$$\begin{aligned} H(\epsilon, u, f, g) := \int_{\Omega_T} & \frac{1}{2}((y^{u,f,g,\epsilon})^2 + (v^{u,f,g,\epsilon})^2 + \alpha(\chi_{\omega_\epsilon} u)^2) + \frac{\partial v^{u,f,g,\epsilon}}{\partial t} p^{u,f,g,\epsilon} \\ & + \nabla y^{u,f,g,\epsilon} \cdot \nabla p^{u,f,g,\epsilon} - \chi_{\omega_\epsilon} u p^{u,f,g,\epsilon} + \frac{\partial y^{u,f,g,\epsilon}}{\partial t} w^{u,f,g,\epsilon} - v^{u,f,g,\epsilon} w^{u,f,g,\epsilon} \, dxdt \\ & + \int_\Omega (y^{u,f,g,\epsilon}(x,0) - f \circ \mathbf{T}_\epsilon) w^{u,f,g,\epsilon}(x,0) + (v^{u,f,g,\epsilon}(x,0) - g \circ \mathbf{T}_\epsilon) p^{u,f,g,\epsilon}(x,0) dx, \end{aligned} \tag{3.8}$$

where

$$H(\epsilon, u, f, g) := \tilde{H}(\epsilon, u, f, g, y^{u,f,g,\epsilon}, v^{u,f,g,\epsilon}, p^{u,f,g,\epsilon}, w^{u,f,g,\epsilon})$$

and $\lambda$ is a small positive number with $\mathbf{T}_\epsilon := \mathrm{id} + \epsilon$.

The following definition will be important.

**Definition 3.3.** For $\epsilon \geq 0$ and given $(u, f, g) \in U \times K_1 \times K_2$, we define the averaged adjoint equations associated with $y^{u,f,g,\epsilon}$ and $y^{u,f,g,\omega}$; $v^{u,f,g,\epsilon}$ and $v^{u,f,g,\omega}$ as: find $p^{u,f,g,\epsilon} \in L^2(H_0^1(\Omega)), w^{u,f,g,\epsilon} \in L^2(L^2(\Omega))$ such that

$$\int_0^1 \partial_y \tilde{H}\Big(\epsilon, u, f, g, s y^{u,f,g,\epsilon} + (1-s) y^{u,f,g,\omega}, v^{u,f,g,\epsilon}, p^{u,f,g,\epsilon}, w^{u,f,g,\epsilon}\Big)(\phi) ds = 0, \tag{3.9}$$

for all $\phi \in L^2(H_0^1(\Omega))$, and

$$\int_0^1 \partial_v \tilde{H}\Big(\epsilon, u, f, g, y^{u,f,g,\epsilon}, s v^{u,f,g,y,\epsilon} + (1-s) v^{u,f,g,\omega}, p^{u,f,g,\epsilon}, w^{u,f,g,\epsilon}\Big)(\psi) ds = 0, \tag{3.10}$$

for all $\psi \in L^2(L^2(\Omega))$, where $\partial_y \tilde{H}$ and $\partial_v \tilde{H}$ denote the partial derivatives of $\tilde{H}$ with respect to $y$ and $v$, respectively.

The following lemma will be important in the proof of the theorem that follows.

**Lemma 3.2.** *The averaged adjoint equations* (3.9) *and* (3.10) *associated with* $y^{u,f,g,\epsilon}$ *and* $y^{u,f,g,\omega}$*;* $v^{u,f,g,\epsilon}$ *and* $v^{u,f,g,\omega}$ *are equivalently given by*

$$\begin{aligned} &\int_{\Omega_T} -\phi \frac{\partial w^{u,f,g,\epsilon}}{\partial t} \, dxdt + \int_{\Omega_T} \nabla \phi \cdot \nabla p^{u,f,g,\epsilon} \, dxdt \\ &= -\int_{\Omega_T} \frac{1}{2}(y^{u,f,g,\epsilon} + y^{u,f,g,\omega}) \phi \, dxdt, \text{ for all } \phi \in L^2(H_0^1(\Omega)) \end{aligned} \tag{3.11}$$

*and*

$$\int_{\Omega_T} -\psi \frac{\partial p^{u,f,g,\epsilon}}{\partial t} - \psi w^{u,f,g,\epsilon} \, dxdt = -\int_{\Omega_T} \frac{1}{2}(v^{u,f,g,\epsilon} + v^{u,f,g,\omega}) \psi \, dxdt, \tag{3.12}$$

*for all* $\psi \in L^2(L^2(\Omega))$*, respectively.*

*Proof.* We prove (3.11) as follows. Since

$$y^{u,f,g,\epsilon} \mapsto \tilde{H}(\epsilon, u, f, g, y^{u,f,g,\epsilon}, v^{u,f,g,\epsilon}, p^{u,f,g,\epsilon}, w^{u,f,g,\epsilon})$$

is affine, $\tilde{H}(\epsilon, u, f, g, y^{u,f,g,\epsilon}, v^{u,f,g,\epsilon}, p^{u,f,g,\epsilon}, w^{u,f,g,\epsilon})$ is Gâteaux differentiable with respect to $y$, see e.g., [22, p.200]. Computing $\partial_y \tilde{H}$ from (3.8), using the Gâteaux derivative determined as a directional derivative in the direction $\phi$, we have

$$\begin{aligned} \partial_y H(\epsilon, u, f, g) = \int_{\Omega_T} \left( \frac{y^{u,f,g,\epsilon} + y^{u,f,g,\omega}}{2} \right) \phi \; dxdt + \int_{\Omega_T} \frac{\partial \phi}{\partial t} w^{u,f,g,\epsilon} \\ + \nabla \phi \cdot \nabla p^{u,f,g,\epsilon} \; dxdt + \int_{\Omega} \phi(x,0) w^{u,f,g,\epsilon}(x,0) \; dx, \end{aligned} \tag{3.13}$$

for all $\phi \in L^2(H_0^1(\Omega))$. Since (3.9) holds, substituting (3.13) in (3.9) gives

$$\begin{aligned} 0 = \int_0^1 \Bigg( \int_{\Omega_T} \left( \frac{y^{u,f,g,\epsilon} + y^{u,f,g,\omega}}{2} \right) \phi \; dxdt + \int_{\Omega_T} \frac{\partial \phi}{\partial t} w^{u,f,g,\epsilon} \\ + \nabla \phi \cdot \nabla p^{u,f,g,\epsilon} \; dxdt + \int_{\Omega} \phi(x,0) w^{u,f,g,\epsilon}(x,0) \; dx \Bigg) \; ds, \end{aligned} \tag{3.14}$$

for all $\phi \in L^2(H_0^1(\Omega))$.

From (3.14), we must have

$$\begin{aligned} &\int_{\Omega_T} \frac{\partial \phi}{\partial t} w^{u,f,g,\epsilon} + \nabla \phi \cdot \nabla p^{u,f,g,\epsilon} \; dxdt + \int_{\Omega} \phi(x,0) w^{u,f,g,\epsilon}(x,0) \; dx \\ &= - \int_{\Omega_T} \frac{1}{2} (y^{u,f,g,\epsilon} + y^{u,f,g,\omega}) \phi \; dxdt, \text{ for all } \phi \in L^2(H_0^1(\Omega)). \end{aligned} \tag{3.15}$$

Integrating the first term of (3.15) by partial integration with respect to $t$, gives

$$\begin{aligned} &\int_{\Omega_T} -\phi \frac{\partial w^{u,f,g,\epsilon}}{\partial t} \; dxdt + \int_{\Omega} \phi(x,T) w^{u,f,g,\epsilon}(x,T) \; dx - \int_{\Omega} \phi(x,0) w^{u,f,g,\epsilon}(x,0) \; dx \\ &+ \int_{\Omega_T} \nabla \phi \cdot \nabla p^{u,h,\epsilon} \; dxdt + \int_{\Omega} \varphi(x,0) w^{u,f,g,\epsilon}(x,0) \; dx \\ &= - \int_{\Omega_T} \frac{1}{2} (y^{u,f,g,\epsilon} + y^{u,f,g,\omega}) \varphi \; dxdt, \text{ for all } \phi \in L^2(H_0^1(\Omega)). \end{aligned} \tag{3.16}$$

Substituting $w^{u,f,g,\epsilon}(x,T) = 0$ in (3.16), the averaged adjoint equation associated with $y^{u,f,g,\epsilon}$ and $y^{u,f,g,\omega}$ becomes

$$\begin{aligned} &\int_{\Omega_T} -\phi \frac{\partial w^{u,f,g,\epsilon}}{\partial t} \; dxdt + \int_{\Omega_T} \nabla \phi \cdot \nabla p^{u,f,g,\epsilon} \; dxdt \\ &= - \int_{\Omega_T} \frac{1}{2} (y^{u,f,g,\epsilon} + y^{u,f,g,\omega}) \phi \; dxdt, \text{ for all } \phi \in L^2(H_0^1(\Omega)). \end{aligned}$$

The proof of (3.12) follows similar arguments in (3.11). □

In the following theorem, the topological derivative of $J_2$ is derived. Note that for simplicity, for $\omega \in E(\Omega)$, $f \in K_1$ and $g \in K_2$. The notations:

$$\overline{y}^{f,g,\omega} := y^{\overline{u}^{f,g,\omega}, f,g,\omega}, \overline{v}^{f,g,\omega} := v^{\overline{u}^{f,g,\omega}, f,g,\omega}, \overline{p}^{f,g,\omega} := p^{\overline{u}^{f,g,\omega}, f,g,\omega}$$

and

$$\overline{w}^{f,g,\omega} := w^{\overline{u}^{f,g,\omega}, f,g,\omega}$$

are used.

**Theorem 3.3.** *Let $\omega \in E(\Omega)$ be open. Let the assumption given in subsection 3.1 hold at $\eta_0 \in \Omega \setminus \partial\omega$. Then the topological derivative of $\omega \mapsto J_2(\omega)$ at $\omega$ in $\eta_0$ is given by*

$$\mathfrak{T}J_2(\omega)(\eta_0) = \max_{(f,g)\in\mathfrak{X}_2(\omega)} \begin{cases} -\int_0^T \overline{u}(\eta_0,s)\overline{p}^{f,g,\omega}(\eta_0,s)ds \text{ if } \eta_0 \in \omega, \\ \\ \int_0^T \overline{u}(\eta_0,s)\overline{p}^{f,g,\omega}(\eta_0,s)ds \text{ if } \eta_0 \in \Omega \setminus \overline{\omega}, \end{cases}$$

*where the adjoint $(\overline{p}^{f,g,\omega}, \overline{w}^{f,g,\omega})$ with $\overline{p}^{f,g,\omega} \in C([0,T] \times \overline{B}_\delta(\eta_0))$ satisfies*

$$\begin{aligned} \frac{\partial \overline{p}^{f,g,\omega}}{\partial t} - \overline{w}^{f,g,\omega} &= -\overline{v}^{f,g,\omega}, \ (x,t) \in \Omega \times (0,T], \\ \frac{\partial \overline{w}^{f,g,\omega}}{\partial t} - \triangle \overline{p}^{f,g,\omega} &= -\overline{y}^{f,g,\omega}, \ (x,t) \in \Omega \times (0,T], \\ \overline{p}^{f,g,\omega}(x,T) = 0, \ \overline{w}^{f,g,\omega}(x,T) &= 0, \ x \in \Omega, \\ \overline{p}^{f,g,\omega} &= 0, \ (x,t) \in \partial\Omega \times (0,T]. \end{aligned}$$

We begin by recalling an important lemma before proving Theorem 3.3. Let $H : [0,\lambda] \times U_{ad} \times K_1 \times K_2 \to \mathbb{R}$ be a function. Then, we define the max-min function $h : [0,\lambda] \to \mathbb{R}$ by

$$h(\epsilon) := \sup_{f\in K_1, g\in K_2} \inf_{u\in U_{ad}} H(\epsilon, u, f, g).$$

In the following lemma, we seek to find out sufficient conditions for the existence of the limit

$$\frac{d}{d\ell} h(0^+) := \lim_{\epsilon \searrow 0^+} \frac{h(\epsilon) - h(0)}{\ell(0)},$$

for any function $\ell : [0,\lambda] \to \mathbb{R}$ such that $\ell(\epsilon) > 0$ for $\epsilon \in (0,\lambda]$, and $\ell(0) = 0$.

**Lemma 3.4.** *Assume that the following hypotheses hold.*

(H0) *The problem*

$$\inf_{u\in U_{ad}} H(\epsilon, u, f, g)$$

*admits a unique optimal solution $\overline{u}$.*

(H1) *The set of maximizers*

$$\mathfrak{X}_2(\omega) := \{(f,g) : \sup_{f\in K_1, g\in K_2} \inf_{u\in U_{ad}} H(\epsilon, u, f, g) = \inf_{u\in U_{ad}} H(\epsilon, u^{\epsilon,f,g}, f, g)\}$$

*is nonempty for all $\epsilon \in [0,\lambda]$.*

(H2) *For all $f \in K_1, g \in K_2$ and $\epsilon \in [0,\lambda]$, the partial derivatives*

$$\lim_{\epsilon \searrow 0} \frac{H(\epsilon, u^{\epsilon,f,g}, f, g) - H(0, u^{\epsilon,f,g}, f, g)}{\ell(\epsilon)}$$

*and*

$$\lim_{\epsilon \searrow 0} \frac{H(\epsilon, u^{0,f,g}, f, g) - H(0, u^{0,f,g}, f, g)}{\ell(\epsilon)}$$

*exist and are equal.*

(H3) *For all $\epsilon_n \in [0,\lambda]$ and $(f_n, g_n) \in \mathfrak{X}_2(\omega_n)$, there exist subsequences $\{\epsilon_{n_k}\}$ and $\{f_{n_k}, g_{n_k}\}$ with $f_{n_k} \rightharpoonup f$ in $H_0^1(\Omega)$ and $g_{n_k} \rightharpoonup g$ in $L^2(\Omega)$ as $k \to \infty$ and $(f,g) \in \mathfrak{X}_2(\omega)$, such that*

$$\lim_{k\to\infty} \frac{H(\epsilon_{n_k}, u_{n_k}, f_{n_k}, g_{n_k}) - H(0, u_{n_k}, f_{n_k}, g_{n_k})}{\ell(\epsilon_{n_k})} = \partial_\ell H(0^+, u^{0,f,g}, f, g)$$

*and*

$$\lim_{k\to\infty} \frac{H(\epsilon_{n_k}, u^{f_{n_k}, g_{n_k}, 0}, f_{n_k}, g_{n_k}) - H(0, u^{f_{n_k}, g_{n_k}, 0}, f_{n_k}, g_{n_k})}{\ell(\epsilon_{n_k})} = \partial_\ell H(0^+, u^{f,g,0}, f, g).$$

*Then, we have*

$$\frac{d}{d\ell} h(\epsilon)|_{\epsilon=0^+} = \max_{(f,g)\in\mathfrak{X}_2(\omega)} \partial_\ell H(0^+, u^{0,f,g}, f, g).$$

*Proof.* We refer to [6, p.524] and [21]. ☐

*Proof of Theorem 3.3.* Since $J_2$ is a max-min function, its topological derivative can be derived by differentiating the function:

$$\sup_{f\in K_1, g\in K_2} \inf_{u\in U_{ad}} H(\epsilon, u, f, g),$$

with respect to $|B_\epsilon(\eta_0)|$. Therefore, we replace $\ell(\epsilon)$ by $|B_\epsilon(\eta_0)|$ and proceed by checking out the hypotheses in Lemma 3.4 as follows. Since $J_1$ and $J_2$ are well-posed (see e.g., [3]), it follows that (H0) and (H1) are satisfied. Next, we check that (H2) and (H3) hold. Using the fundamental theorem of calculus on averaged adjoint equation (3.9) with $\epsilon \geq 0$, we have

$$\begin{aligned} h(\epsilon) :=& \tilde{H}(\epsilon, u, f, g, y^{u,f,g,\epsilon}, v^{u,f,g,\epsilon}, p^{u,f,g,\epsilon}, w^{u,f,g,\epsilon}), \\ =& \tilde{H}(\epsilon, u, f, g, y^{u,f,g,\omega}, v^{u,f,g,\omega}, p^{u,f,g,\epsilon}, w^{u,f,g,\epsilon}). \end{aligned} \tag{3.17}$$

Using (3.17), we deduce that

$$h(0) = \tilde{H}(0, u, f, g, y^{u,f,g,\omega}, v^{u,f,g,\omega}, p^{u,f,g,\epsilon}, w^{u,f,g,\epsilon}). \tag{3.18}$$

From (3.17) and (3.18), we get

$$\begin{aligned} &h(\epsilon) - h(0) \\ &= \tilde{H}(\epsilon, u, f, g, y^{u,f,g,\omega}, v^{u,f,g,\omega}, p^{u,f,g,\epsilon}, w^{u,f,g,\epsilon}) - \tilde{H}(0, u, f, g, y^{u,f,g,\omega}, v^{u,f,g,\omega}, p^{u,f,g,\epsilon}, w^{u,f,g,\epsilon}). \end{aligned} \tag{3.19}$$

Using the notations

$$\overline{u}_n := u^{f_n, g_n, \omega_{\epsilon_n}}, \ \overline{y}^{f_n,g_n,\omega} := y^{\overline{u}_n, f_n, g_n, \omega}, \ \overline{v}^{f_n,g_n,\omega} := v^{\overline{u}_n, f_n, g_n, \omega},$$

$$\overline{p}^{f_n,g_n,\epsilon_n} := p^{\overline{u}_n, f_n, g_n,\ \omega_{\epsilon_n}}, \ \overline{w}^{f_n,g_n,\epsilon_n} := w^{\overline{u}_n, f_n,\ g_n, \omega_{\epsilon_n}}$$

and

$$H(\epsilon_n, \overline{u}_n, f_n, g_n) := \tilde{H}(\epsilon_n, \overline{u}_n, f_n, g_n, \overline{y}^{f_n,g_n,\omega}, \overline{v}^{f_n,g_n,\omega}, \overline{p}^{f_n,g_n,\epsilon_n}, \overline{w}^{f_n,g_n,\epsilon_n}),$$

we obtain

$$\begin{aligned} H(\epsilon_n, \overline{u}_n, f_n, g_n) \overset{(3.17)}{=} &\int_{\Omega_T} \frac{1}{2}(\overline{y}^{f_n,g_n,\omega})^2 + \frac{1}{2}(\overline{v}^{f_n,g_n,\omega})^2 + \frac{\alpha}{2}(\chi_{\omega_{\epsilon_n}} \overline{u}_n)^2 + \frac{\partial \overline{v}^{f_n,g_n,\omega}}{\partial t} \overline{p}^{f_n,g_n,\epsilon_n} \\ &+ \nabla \overline{y}^{f_n,g_n,\omega} \cdot \nabla \overline{p}^{f_n,g_n,\epsilon_n} - \chi_{\omega_{\epsilon_n}} \overline{u}_n \overline{p}^{f_n,g_n,\epsilon_n} + \frac{\partial \overline{y}^{f_n,g_n,\omega}}{\partial t} \overline{w}^{f_n,g_n,\epsilon_n} \\ &- \overline{v}^{f_n,g_n,\omega} \overline{w}^{f_n,g_n,\epsilon_n} \, dxdt + \int_\Omega (\overline{y}^{f_n,g_n,\omega}(x,0) - f_n \circ \mathbf{T}_{\epsilon_n}) \overline{w}^{f_n,g_n,\epsilon_n}(x,0) \\ &+ (\overline{v}^{f_n,g_n,\omega}(x,0) - g_n \circ \mathbf{T}_{\epsilon_n}) \overline{p}^{f_n,g_n,\epsilon_n}(x,0) \, dx, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
H(0,\overline{u}_n,f_n,g_n) \overset{(3.18)}{=} & \int_{\Omega_T} \frac{1}{2}(\overline{y}^{f_n,g_n,\omega})^2 + \frac{1}{2}(\overline{v}^{f_n,g_n,\omega})^2 + \frac{\alpha}{2}(\chi_{\omega_{\epsilon_n}}\overline{u}_n)^2 + \frac{\partial \overline{v}^{f_n,g_n,\omega}}{\partial t}\overline{p}^{f_n,g_n,\epsilon_n} \\
& + \nabla \overline{y}^{f_n,g_n,\omega}\cdot\nabla\overline{p}^{f_n,g_n,\epsilon_n} - \chi_\omega \overline{u}_n\overline{p}^{f_n,g_n,\epsilon_n} + \frac{\partial \overline{y}^{f_n,g_n,\omega}}{\partial t}\overline{w}^{f_n,g_n,\epsilon_n} \\
& - \overline{v}^{f_n,g_n,\omega}\overline{w}^{f_n,g_n,\epsilon_n}\ dxdt + \int_\Omega (\overline{y}^{f_n,g_n,\omega}(x,0) - f_n\circ\mathbf{T}_{\epsilon_n})\overline{w}^{f_n,g_n,\epsilon_n}(x,0) \qquad (3.21)\\
& + (\overline{v}^{f_n,g_n,\omega}(x,0) - g_n\circ\mathbf{T}_{\epsilon_n})\overline{p}^{f_n,g_n,\epsilon_n}(x,0)\ dx.
\end{aligned}$$

Suppose, without loss of generality, $\eta_0\in\omega$ and $\omega_\epsilon := \omega\setminus\overline{B}_\epsilon(\eta_0)$. Then, subtracting (3.21) from (3.20) and dividing through by $|B_{\epsilon_n}(\eta_0)|$, we obtain

$$\begin{aligned}
&\frac{H(\epsilon_n,\overline{u}_n,f_n,g_n) - H(0,\overline{u}_n,f_n,g_n)}{|B_{\epsilon_n}(\eta_0)|} \\
&= -\frac{1}{|B_{\epsilon_n}(\eta_0)|}\int_0^T\int_{B_{\epsilon_n}(\eta_0)} \overline{u}_n\overline{p}^{f_n,g_n,\epsilon_n}\ dxdt, \\
&= -\frac{1}{|B_{\epsilon_n}(\eta_0)|}\int_0^T\int_{B_{\epsilon_n}(\eta_0)} \overline{u}_n(\overline{p}^{f_n,g_n,\epsilon_n} - \overline{p}^{f,g,\omega}) - (\overline{u}_n - \overline{u})\overline{p}^{f,g,\omega} - \overline{u}\overline{p}^{f,g,\omega}\ dxdt. \qquad (3.22)
\end{aligned}$$

We estimate the terms in (3.22) as follows. By Hölder's inequality, for any $n\in\{0\}\cup\mathbb{N}$, we have for the second term in (3.22)

$$\begin{aligned}
&\frac{1}{|B_{\epsilon_n}(\eta_0)|}\left|\int_0^T\int_{B_{\epsilon_n}(\eta_0)} (\overline{u}_n - \overline{u})\overline{p}^{f,g,\omega}\ dxdt\right| \qquad (3.23)\\
&\leq \frac{1}{|B_{\epsilon_n}(\eta_0)|}||\overline{u}_n - \overline{u}||_{L^p(0,T;C(\overline{B}_\delta(\eta_0)))}||\overline{p}^{f,g,\omega}||_{L^q(0,T;C(\overline{B}_\delta(\eta_0)))}. \qquad (3.24)
\end{aligned}$$

If $p=1$, then $q=\infty$. Hence, right-hand side of (3.24) is bounded by

$$\frac{1}{|B_{\epsilon_n}(\eta_0)|}||\overline{u}_n - \overline{u}||_{L^1(0,T;C(\overline{B}_\delta(\eta_0)))}||\overline{p}^{f,g,\omega}||_{C([0,T]\times\overline{B}_\delta(\eta_0))}. \qquad (3.25)$$

Similarly, by Hölder's inequality, we have for the first term in (3.22)

$$\begin{aligned}
&\frac{1}{|B_{\epsilon_n}(\eta_0)|}\left|\int_0^T\int_{B_{\epsilon_n}(\eta_0)} \overline{u}_n(\overline{p}^{f_n,g_n,\epsilon_n} - \overline{p}^{f,g,\omega})\ dxdt\right| \\
&\leq \frac{1}{|B_{\epsilon_n}(\eta_0)|}||\overline{u}_n||_{L^1(0,T;C(\overline{B}_\delta(\eta_0)))}||\overline{p}^{f_n,g_n,\epsilon_n} - \overline{p}^{f,g,\omega}||_{C([0,T]\times\overline{B}_\delta(\eta_0))}. \qquad (3.26)
\end{aligned}$$

We estimate the last term in (3.22), for any $n\in\{0\}\cup\mathbb{N}$, as follows. Since

$$\frac{1}{|B_{\epsilon_n}(\eta_0)|}\int_0^T\int_{B_{\epsilon_n}(\eta_0)} \overline{u}(x,t)\overline{p}^{f,g,\omega}(x,t)\ dxdt = \frac{1}{|B_{\epsilon_n}(\eta_0)|}\int_{(0,T]\times B_{\epsilon_n}(\eta_0)} \overline{u}(x,t)\overline{p}^{f,g,\omega}(x,t)\ dxdt, \quad (3.27)$$

by interchanging the order of integration in the right-hand side of (3.27), we have

$$\frac{1}{|B_{\epsilon_n}(\eta_0)|}\int_{(0,T]\times B_{\epsilon_n}(\eta_0)} \overline{u}(x,t)\overline{p}^{f,g,\omega}(x,t)\ dxdt = \frac{1}{|B_{\epsilon_n}(\eta_0)|}\int_{B_{\epsilon_n}(\eta_0)}\int_0^T \overline{u}(x,t)\overline{p}^{f,g,\omega}(x,t)\ dt\ dx.$$

Thus,

$$\frac{1}{|B_{\epsilon_n}(\eta_0)|}\left|\int_{B_{\epsilon_n}(\eta_0)}\int_0^T \overline{u}(x,t)\overline{p}^{f,g,\omega}(x,t)\ dtdx\right| \leq \frac{1}{|B_{\epsilon_n}(\eta_0)|}\int_{B_{\epsilon_n}(\eta_0)}\left|\int_0^T \overline{u}(x,t)\overline{p}^{f,g,\omega}(x,t)\ dt\right| dx.$$

Since $x \mapsto \int_0^T \overline{u}(x,t)\overline{p}^{f,g,\omega}(x,t)\ dt$ is continuous in the neighbourhood of $\eta_0$, we have

$$\frac{1}{|B_{\epsilon_n}(\eta_0)|}\left|\int_{B_{\epsilon_n}(\eta_0)}\int_0^T \overline{u}(x,t)\overline{p}^{f,g,\omega}(x,t)\ dtdx\right| \leq \left|\int_0^T \overline{u}(\eta_0,t)\overline{p}^{f,g,\omega}(\eta_0,t)\ dt\right|. \tag{3.28}$$

Using $\eta_0 \in \omega$ in (3.4), (3.5), and (3.7) and passing to the limits in (3.25), (3.26), and (3.28); we see that the right-hand side of (3.22) converges to $-\int_0^T \overline{u}(\eta_0,t)\overline{p}^{f,g,\omega}(\eta_0,t)\ dt$. Therefore, $h'(0)$ exists and is given by

$$\lim_{n\to\infty}\frac{H(\epsilon_n,\overline{u}_n,f_n,g_n)-H(0,\overline{u}_n,f_n,g_n)}{|B_{\epsilon_n}(\eta_0)|} = -\int_0^T \overline{u}(\eta_0,t)\overline{p}^{f,g,\omega}(\eta_0,t)\ dt. \tag{3.29}$$

Suppose that $\overline{u}_{n,0} := \overline{u}^{f_n,g_n,0}$. Then similarly, modifying $\overline{u}_n$ as $\overline{u}_{n,0}$, we obtain

$$\lim_{n\to\infty}\frac{H(\epsilon_n,\overline{u}_{n,0},f_n,g_n)-H(0,\overline{u}_{n,0},f_n,g_n)}{|B_{\epsilon_n}(\eta_0)|} = -\int_0^T \overline{u}(\eta_0,t)\overline{p}^{f,g,\omega}(\eta_0,t)\ dt. \tag{3.30}$$

Suppose that $\{f_n\}$ and $\{g_n\}$ are constant sequences. Then, it is clearly seen that $H(\epsilon_n,\overline{u}_n,f_n,g_n) - H(0,\overline{u}_n,f_n,g_n)$ in (3.29) and $H(\epsilon_n,\overline{u}_{n,0},f_n,g_n) - H(0,\overline{u}_{n,0},f_n,g_n)$ in (3.30) are equal. Hence, (H2) is satisfied. Note that for every null sequence $\{\epsilon_n\}$ in $[0,\lambda]$ and every sequence $\{f_n,g_n\}$ with $(f_n,g_n)\in\mathfrak{X}_2(\omega_{\epsilon_n})$, we can find a subsequence $\{f_{n_k},g_{n_k}\}$, such that $f_{n_k}\rightharpoonup f$ in $H_0^1(\Omega)$ and $g_{n_k}\rightharpoonup g$ in $L^2(\Omega)$ as $k\to\infty$, where $(f,g)\in\mathfrak{X}_2(\omega)$. Thus, we obtain the left-hand side of (3.29) and (3.30) as $\partial_\epsilon H(0^+,\overline{u}^{0,f,g},f,g)$ and $\partial_\epsilon H(0^+,\overline{u}^{f,g,0},f,g)$, respectively. Hence, (H3) is satisfied. $\square$

As a consequence of Theorem 3.3, we obtain the directional derivative of $J_1$.

**Corollary 3.5.** *Let the assumption given in subsection 3.1 hold. Let $(f,g)\in H_0^1(\Omega)\times L^2(\Omega)$ be given. Then the following derivative of $\omega\mapsto J_1(\omega,f,g)$ at $\omega$ in $\eta_0$ is given by*

$$\mathfrak{T}J_1(\omega,f,g)(\eta_0) = \begin{cases} -\int_0^T \overline{u}^{f,g,\omega}(\eta_0,s)\overline{p}^{f,g,\omega}(\eta_0,s)ds \text{ if } \eta_0\in\omega, \\ \\ \int_0^T \overline{u}^{f,g,\omega}(\eta_0,s)\overline{p}^{f,g,\omega}(\eta_0,s)ds \text{ if } \eta_0\in\Omega\setminus\overline{\omega}, \end{cases} \tag{3.31}$$

*where the adjoint $(p^{\overline{u},f,g,\omega}, w^{\overline{u},f,g,\omega})$ satisfies the adjoint equations:*

$$\begin{aligned} \frac{\partial p^{\overline{u},f,g,\omega}}{\partial t} - w^{\overline{u},f,g,\omega} &= -v^{\overline{u},f,g,\omega},\ (x,t)\in\Omega\times(0,T], \\ \frac{\partial w^{\overline{u},f,g,\omega}}{\partial t} - \triangle p^{\overline{u},f,g,\omega} &= -y^{\overline{u},f,g,\omega},\ (x,t)\in\Omega\times(0,T], \\ p^{\overline{u},f,g,\omega}(x,T) = 0,\ \ w^{\overline{u},f,g,\omega}(x,T) &= 0,\ x\in\Omega, \\ p^{\overline{u},f,g,\omega} &= 0,\ (x,t)\in\partial\Omega\times(0,T]. \end{aligned}$$

*Proof.* The proof of (3.31) follows from the hypotheses of Theorem 3.3. $\square$

Next, we discuss the discretization of the optimization problems, the homotopy continuation algorithm, and numerical solutions for different cases of initial conditions $f$ and $g$.

## 4. Numerical Examples

4.1. **State Equation.** We discretize the system (3.1) as follows. Since we are solving for $v$ and $y$ forward in time, it follows that the time increment is positive. Thus, using the second equation in (3.1) and taking a weighted finite difference in time, we obtain:

$$\frac{v^{k+1}-v^k}{\triangle t} = \theta\triangle y^{k+1} + (1-\theta)\triangle y^k + \chi_\omega u^k, \tag{4.1}$$

where the parameter $\theta \in [0,1]$ and $\triangle t$ is the step size in time. Taking finite elements in space in (4.1), yields:

$$\frac{M_h\mathbf{v}_h^{k+1} - M_h\mathbf{v}_h^k}{\triangle t} = -\theta S_h\mathbf{y}_h^{k+1} - (1-\theta)S_h\mathbf{y}_h^k + M_h\chi_\omega\mathbf{u}_h^k, \tag{4.2}$$

where

$$\begin{aligned}&\mathbf{y}_h = (y_1, y_2, \ldots, y_N)^\top, \mathbf{v}_h = (v_1, v_2, \ldots, v_N)^\top,\\ &\mathbf{u}_h = (u_1, u_2, \ldots, u_N)^\top, M_h, S_h \in \mathbb{R}^{N\times N}.\end{aligned} \tag{4.3}$$

Here, $M_h$ and $S_h$ represent mass and stiffness matrices of mesh size $h$, respectively, see e.g., [5]. Simplifying (4.2) gives the system:

$$\begin{aligned}&M_h\mathbf{v}_h^{k+1} = M_h\mathbf{v}_h^k - \triangle t\theta S_h\mathbf{y}_h^{k+1} - \triangle t(1-\theta)S_h\mathbf{y}_h^k + \triangle t M_h\chi_\omega\mathbf{u}_h^k,\\ &k = 1, 2, \ldots, N-1.\end{aligned} \tag{4.4}$$

Substituting $k = 0$ in (4.1), we obtain

$$\frac{v^1 - v^0}{\triangle t} = \theta\triangle y^1 + (1-\theta)\triangle y^0 + \chi_\omega u^0. \tag{4.5}$$

Taking finite elements in space in (4.5) and simplifying, we obtain:

$$M_h\mathbf{v}_h^1 = M_h\mathbf{v}_h^0 - \triangle t\theta S_h\mathbf{y}_h^1 - \triangle t(1-\theta)S_h\mathbf{y}_h^0 + \triangle t M_h\chi_\omega\mathbf{u}_h^0. \tag{4.6}$$

Using (4.6) and the third equation in (3.1), we have

$$M_h\mathbf{v}_h^1 = M_h\mathbf{g}_h - \triangle t\theta S_h\mathbf{y}_h^1 - \triangle t(1-\theta)S_h\mathbf{f}_h + \triangle t M_h\chi_\omega\mathbf{u}_h^0. \tag{4.7}$$

Similarly, the first equation in (3.1) gives

$$\mathbf{y}_h^{k+1} = \mathbf{y}_h^k + \triangle t\mathbf{v}_h^k, k = 1, 2, \ldots, N-1, \tag{4.8}$$

and for $k = 0$

$$\mathbf{y}_h^1 = \mathbf{f}_h + \triangle t\mathbf{g}_h. \tag{4.9}$$

We formulate the discrete version of the adjoint in the following subsection.

4.2. **Adjoint Equation.** We discretize the adjoint equation (3.2) as follows. Since we are solving for $w$ and $p$ backward in time, the time increment is negative. Using the second equation in (3.2) and taking a weighted finite difference in time, we get:

$$\frac{w^{k-1} - w^k}{\triangle t} = \theta\triangle p^{k-1} + (1-\theta)\triangle p^k - y^k. \tag{4.10}$$

Taking finite elements in space in (4.10), we obtain:

$$\frac{M_h\mathbf{w}_h^{k-1} - M_h\mathbf{w}_h^k}{\triangle t} = -\theta S_h\mathbf{p}_h^{k-1} - (1-\theta)S_h\mathbf{p}_h^k - M_h\mathbf{y}_h^k, \tag{4.11}$$

where $\mathbf{p}_h = (p_1, p_2, \ldots, p_N)^\top, \mathbf{w}_h = (w_1, w_2, \ldots, w_N)^\top$.
Simplifying (4.11) gives the $\theta-$scheme:

$$M_h\mathbf{w}_h^{k-1} = M_h\mathbf{w}_h^k - \triangle t\theta S_h\mathbf{p}_h^{k-1} - \triangle t(1-\theta)S_h\mathbf{p}_h^k - \triangle t M_h\mathbf{y}_h^k, k = 1, \ldots, N-1. \tag{4.12}$$

Substituting $k = N$ in (4.10) gives

$$\frac{\mathbf{w}_h^{N-1} - \mathbf{w}_h^N}{\triangle t} = \theta\triangle\mathbf{p}_h^{N-1} + (1-\theta)\triangle\mathbf{p}_h^N - \mathbf{y}_h^N. \tag{4.13}$$

Taking finite elements in space in (4.13) and simplifying, we obtain:

$$M_h \mathbf{w}_h^{N-1} = M_h \mathbf{w}_h^N - \triangle t\theta S_h \mathbf{p}_h^{N-1} - \triangle t(1-\theta) S_h \mathbf{p}_h^N - \triangle t M_h \mathbf{y}_h^N. \tag{4.14}$$

Using (4.14) and the third equation in (3.2), we have

$$M_h \mathbf{w}_h^{N-1} = -\triangle t\theta S_h \mathbf{p}_h^{N-1} - \triangle t M_h \mathbf{y}_h^N. \tag{4.15}$$

Similarly, the first equation in (3.2) gives

$$\mathbf{p}_h^{k-1} = \mathbf{p}_h^k + \triangle t \mathbf{w}_h^k - \triangle t \mathbf{v}_h^k, k = 1, 2, \ldots, N-1, \tag{4.16}$$

and for $k = N$

$$\mathbf{p}_h^{N-1} = -\mathbf{v}_h^N. \tag{4.17}$$

We formulate the discrete version of the linear-quadratic optimization problems in the following subsection.

4.3. **Functionals $J_1$ and $J_2$.** To compute $J_1$ and $J_2$, we discretize the cost functional using linear finite elements. Substituting the approximations

$$y_h(x,t) = \sum_{i=1}^N y_i(t)\Phi_i(x), v_h(x,t) = \sum_{i=1}^N v_i(t)\Phi_i(x), \chi_\omega u_h(x,t) = \sum_{i=1}^N \chi_\omega u_i(t)\Phi_i(x),$$

where $\Phi_i, i = 1, 2, \ldots, N$ are linear basis functions and $y_i(t), v_i(t), u_i(t)$,
for $i = 1, 2, \ldots, N$ are real unknowns, in (2.2 ), we find that the discrete cost functional $J_h$ is given by

$$J_h(\omega, \mathbf{u}_h, \mathbf{f}_h, \mathbf{g}_h) = \frac{1}{2}\int_0^T \mathbf{y}_h(t)^\top M_h \mathbf{y}_h(t) + \mathbf{v}_h(t)^\top M_h \mathbf{v}_h(t) + \alpha\chi_\omega \mathbf{u}_h(t)^\top M_h \chi_\omega \mathbf{u}_h(t) dt. \tag{4.18}$$

From (4.18), the discrete functionals $J_{1,h}$ and $J_{2,h}$ are defined by

$$J_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h) = \min_{\mathbf{u}_h \in U_{ad}} J_h(\omega, \mathbf{u}_h, \mathbf{f}_h, \mathbf{g}_h) \tag{4.19}$$

and

$$J_{2,h}(\omega) = \max_{\mathbf{f}_h, \mathbf{g}_h} J_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h), \tag{4.20}$$

respectively. Therefore, the discrete derivatives $\mathfrak{T}J_{1,h}(\omega, f, g)$ and $\mathfrak{T}J_{2,h}(\omega)$ of $J_1(\omega, f, g)$ and $J_2(\omega)$ can be deduced from the first equation in (3.31) as

$$\mathfrak{T}J_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h) = -\int_0^T \mathbf{u}_h(t)^\top \mathbf{p}_h(t)\ dt \tag{4.21}$$

and

$$\mathfrak{T}J_{2,h}(\omega) = \max_{\mathbf{f}_h, \mathbf{g}_h} \mathfrak{T}J_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h), \tag{4.22}$$

respectively.

4.4. **The Homotopy Continuation Algorithm for Optimal Actuator Design.** Here, we use a homotopy continuation method (see, e.g., [2], [1]) to determine an optimal actuator design. The topological derivatives are embedded into this method as follows. The penalty term $\beta(|\omega| - |\omega_d|)^2$, where $\beta > 0$ a penalty parameter, is added to the discrete functionals (4.19)–(4.20) while $2\beta(|\omega| - |\omega_d|)$ is added to (4.21)–(4.22) if $\eta_0 \in \omega$ or subtracted from (4.21)–(4.22) if $\eta_0 \in \Omega \setminus \overline{\omega}$. The penalty terms ensure that the size constraint $|\omega| - |\omega_d| = 0$ and optimal designs are achieved. Thus,

$$J_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h) = J^{LQ}_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h) + J^{\beta}_{1,h}(\omega) \text{ and } J_{2,h}(\omega) = J^{LQ}_{2,h}(\omega) + J^{\beta}_{2,h}(\omega).$$

Choosing the point $\eta_0 \in \omega$ implies that we use the topological derivative

$$\mathfrak{T}J_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h)(\eta_0) = -\int_0^T \overline{u}^{\mathbf{f}_h,\mathbf{g}_h,\omega}(\eta_0, s)\overline{p}^{\mathbf{f}_h,\mathbf{g}_h,\omega}(\eta_0, s)\ ds + 2\beta(|\omega| - |\omega_d|)$$

while choosing $\eta_0 \in \Omega \setminus \overline{\omega}$ implies

$$\mathfrak{T}J_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h)(\eta_0) = \int_0^T \overline{u}^{\mathbf{f}_h,\mathbf{g}_h,\omega}(\eta_0, s)\overline{p}^{\mathbf{f}_h,\mathbf{g}_h,\omega}(\eta_0, s) ds - 2\beta(|\omega| - |\omega_d|).$$

The linear-quadratic (LQ) parts $J^{LQ}_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h)$ and $J^{LQ}_{2,h}(\omega)$ are equal to the right-hand sides of (4.19) and (4.20), respectively, and $J^{\beta}_{n,h}(\omega) = \beta(|\omega| - |\omega_d|)^2, n = 1, 2$ is a quadratic penalty term. Let $d := \mathfrak{T}J_{1,h}(\omega, \mathbf{f}_h, \mathbf{g}_h)$. Then, our version of the homotopy continuation algorithm for solving optimal actuator design problems (2.5)−(2.6) proposed in [2] is presented as follows:

**Algorithm 1** Homotopy continuation algorithm for optimal actuator design

**Require:** $\omega_0 \in E(\Omega), f, g$, tolerance $\varepsilon > 0, k = 0, \epsilon, d_0 := \mathfrak{T}J_{1,h}(\omega_0, \mathbf{f}_h, \mathbf{g}_h)$.
  **while** $\|\omega_{k+1} - \omega_k\|_2 \geq \varepsilon$ **do**
    **if** $J_{1,h}(\omega_{k+1}, \mathbf{f}_h, \mathbf{g}_h) < J_{1,h}(\omega_k, \mathbf{f}_h, \mathbf{g}_h)$ **then**
      $d_k = \mathfrak{T}J_{1,h}(\omega_k, \mathbf{f}_h, \mathbf{g}_h)$
      $\omega_{k+1} = \epsilon d_k \omega_k$
      $k := k + 1$
    **end if**
  **end while**
  **return** optimal actuator design $\omega_{k+1}$

From Algorithm 1, we can remark on the following.

*Remark* 4.1.
(i) To investigate the optimal actuator design using $J_2$, we must replace $J_1$ with $J_2$ in Algorithm 1.
(ii) We represent $\omega_k$ by its centre point so that $\omega_k \in \mathbb{R}^d$. Thus, the distance $\|\omega_{k+1} - \omega_k\|_2$ can be computed from the inner product

$$< \omega_{k+1} - \omega_k, \omega_{k+1} - \omega_k >= \sum_{i=1}^{d} |\omega_{(k+1)_i} - \omega_{k_i}|^2.$$

(iii) The stopping criterion is satisfied when

$$\int_0^T \mathbf{u}_h(t)^\top \mathbf{p}_h(\eta_0, t)\ dt = 2\beta(|\omega| - |\omega_d|),$$

i.e., $\|\omega_{k+1} - \omega_k\|_2 < \varepsilon$. If in addition, $|(|\omega| - |\omega_d|)| < e$ ($e$ the round off error in $|\omega| - |\omega_d|$) is satisfied, then the cost and actuator design are optimal.

(iv) Starting with a small initial value of $\beta$, we implement Algorithm 1. The final actuator obtained is used to initialize the subsequent solve with an increasing value of $\beta$ based on a continuation strategy.

In the following subsection, we implement the schemes (4.4), (4.7)−(4.9) and (4.12), (4.15)−(4.17) for two different cases of initial conditions $f$ and $g$. The numerical examples are project-limited to 1-dimensional optimal actuator designs.

4.5. **Examples.** Algorithm 1 is implemented by setting $\varepsilon = 10^{-6}, |\omega_d| = 0.2$, and initial actuator size to 0.3 .

**Example 4.1.** In this test, we set

$$\begin{aligned} &y(x,0) = x^3 - 1.825x^2 + 0.825x, \quad && 0 \le x \le 1, \\ &v(x,0) = 0, && 0 \le x \le 1, \end{aligned}$$

so that the initial displacement of the dynamics is asymmetric, see Figure 1 (a). Two types of tests, with and without a continuation strategy are implemented. First, we begin with $J_{1,h}$. The results are presented in Figure 1 and Table 1. From Table 1, it is observed that as the penalty parameter value increases, the actuator size decreases until an optimal actuator size of 0.221 is attained.

Table 1. The optimization values for $y(x,0) = x^3 - 1.825x^2 + 0.825x,\ v(x,0) = 0$. Each row is initialized with the final actuator corresponding to the previous one.

| $\beta$ | $J_{1,h}(\omega,f,g)$ | $J^{LQ}_{1,h}(\omega,f,g)$ | $J^{\beta}_{1,h}(\omega)$ | actuator size | iterations |
|---|---|---|---|---|---|
| $10^{-2}$ | 0.0208 | 0.0207 | $7.51 \times 10^{-5}$ | 0.287 | 58 |
| $10^{-1}$ | 0.0257 | 0.0251 | $6.16 \times 10^{-4}$ | 0.279 | 9 |
| 1 | 0.0306 | 0.0302 | $4.48 \times 10^{-4}$ | 0.221 | 8 |

With $J_{2,h}$ setting, the results are presented in Figure 2 and Table 2. From Table 2, for a value of $\beta = 1$, an optimal solution with a smaller cost and error than for $\beta = 10^{-1}$ is obtained. Thus, the optimal actuator size is taken to be 0.2001.

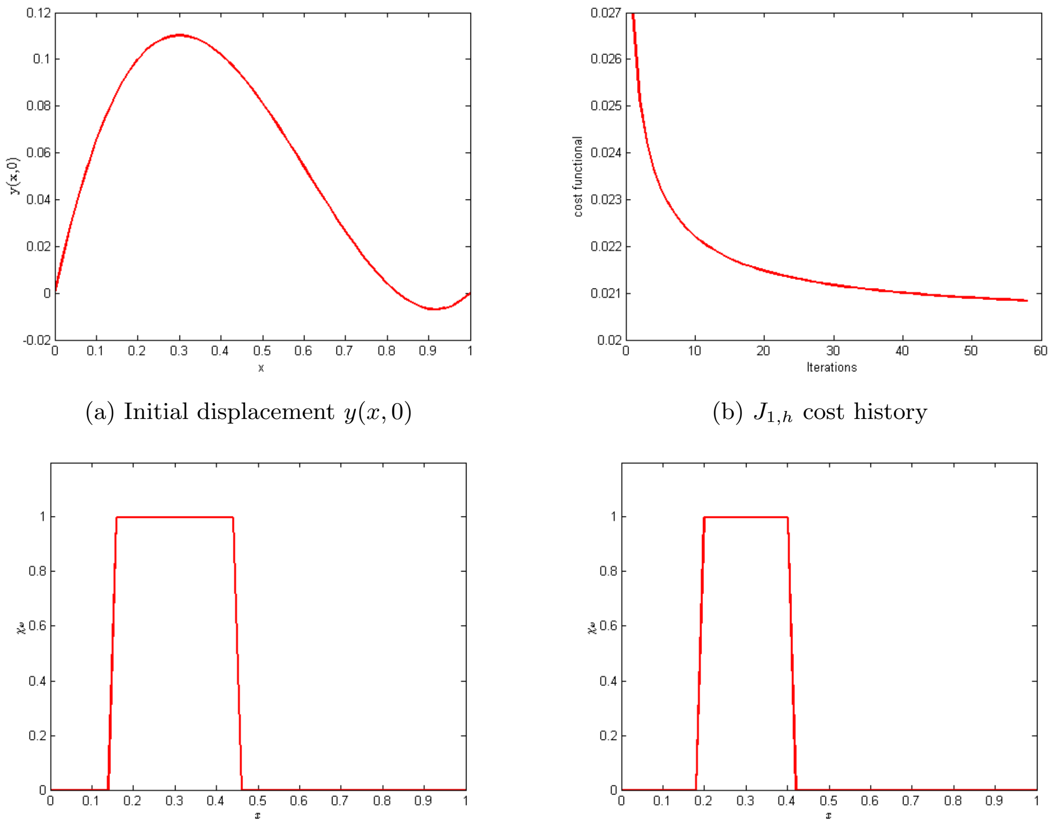
Table 2. The optimization values for $y(x,0) = x^3 - 1.825x^2 + 0.825x,\ v(x,0) = 0$. Each row is initialized with the final actuator corresponding to the previous one.

| $\beta$ | $J_{2,h}(\omega)$ | $J^{LQ}_{2,h}(\omega)$ | $J^{\beta}_{2,h}(\omega)$ | actuator size | iterations |
|---|---|---|---|---|---|
| $10^{-2}$ | 0.1012 | 0.1011 | $5.206 \times 10^{-5}$ | 0.2722 | 16 |
| $10^{-1}$ | 0.1000 | 0.0999 | $6.890 \times 10^{-5}$ | 0.2262 | 22 |
| 1 | 0.0990 | 0.0990 | $1.744 \times 10^{-8}$ | 0.2001 | 38 |

**Example 4.2.** For this test, the initial conditions for the dynamics are set to be

$$\begin{aligned} &y(x,0) = \sin^8(3\pi x), \quad && 0 \le x \le 1, \\ &v(x,0) = 0, && 0 \le x \le 1. \end{aligned}$$

This setting is considered so that the initial displacement has three extrema, see Figure 3 (a). We implemented two types of tests, with and without a continuation strategy. With $J_{1,h}$, the results are depicted in Figure 3 and Table 3. As expected from the symmetry of the initial displacement, we see that the actuator splits into three equally sized-components. Additionally, as the penalty parameter




(a) Initial displacement $y(x,0)$

(b) $J_{1,h}$ cost history

(c) $\beta = 10^{-2}$, without initialization

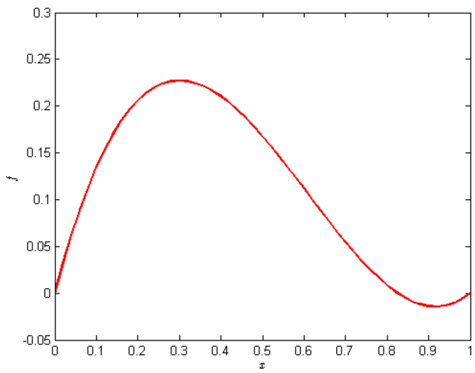
(d) $\beta = 1$

FIGURE 1. (a) Initial displacement $y(x,0) = x^3 - 1.825x^2 + 0.825x$. (b) Decay of total cost $J_{1,h}$. (c) Final actuator for $\beta = 10^{-2}$ was subsequently used in the penalty approach. (d) Optimal actuator for $\beta = 1$, via increasing penalty parameter value.

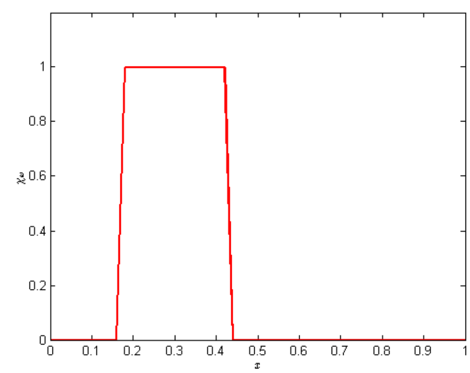
value increases, the actuator size decreases until the desired actuator size of 0.2 is approached. From Table 3, column 3, we see that the LQ part of $J_{1,h}$ tends to a stationary value of 2.40. It is noted that the cost $J_{1,h}$ obtained for the final value of $\beta = 1$ using a continuation approach is 0.0016 smaller than the one without the initialization procedure. Therefore, the optimal actuator size is taken to be 0.1960. Also, from Table 3, it is observed that an actuator size of 0.7802 is obtained after 161 iterations. This is a sub-optimal solution since the size constraint is not satisfied.

TABLE 3. The optimization values for $y(x,0) = \sin^8(3\pi x)$, $v(x,0) = 0$. Each row, except for the last row with $\beta = 1^*$, is initialized with the final actuator corresponding to the previous one.

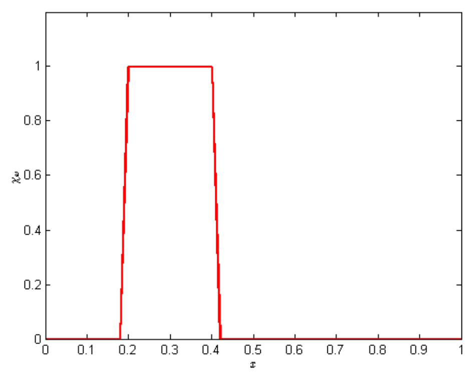
| $\beta$ | $J_{1,h}(\omega,f,g)$ | $J_{1,h}^{LQ}(\omega,f,g)$ | $J_{1,h}^{\beta}(\omega)$ | actuator size | iterations |
|---|---|---|---|---|---|
| $10^{-2}$ | 1.1618 | 1.1517 | $1.0100 \times 10^{-2}$ | 0.7802 | 161 |
| $10^{-1}$ | 2.0520 | 2.0520 | $4.9955 \times 10^{-7}$ | 0.1987 | 48 |
| 1 | 2.3953 | 2.3953 | $3.6317 \times 10^{-5}$ | 0.1960 | 8 |
| $1^*$ | 2.3969 | 2.3969 | $1.0017 \times 10^{-5}$ | 0.1982 | 8 |



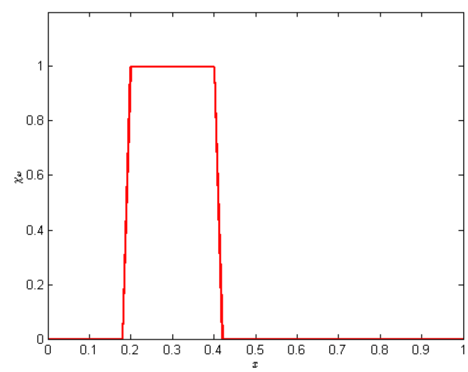

(a) $f$, where $(f, g) \in \mathfrak{X}_2(\omega)$




(b) $\beta = 10^{-2}$, without initialization




(c) $\beta = 10^{-1}$



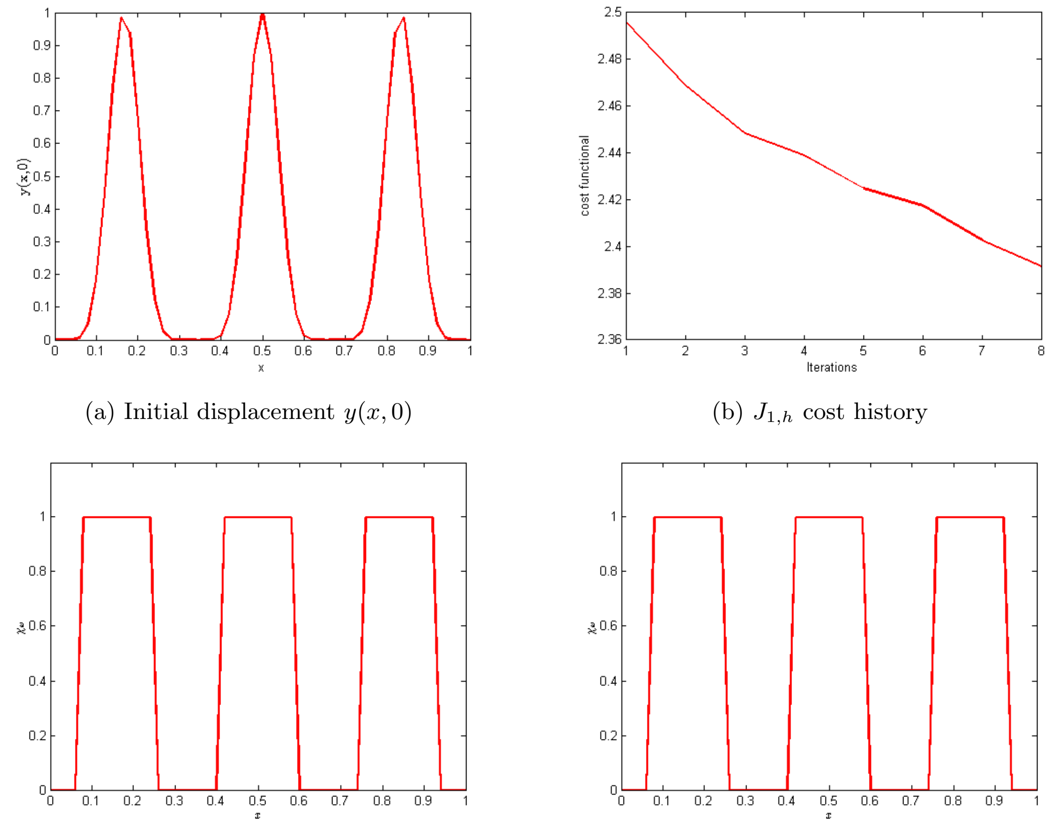

(d) $\beta = 1$

FIGURE 2. (a) The normalized initial displacement $f$. (b) Final actuator for $\beta = 10^{-2}$ was subsequently used in the penalty approach. (c) Optimal actuator for $\beta = 10^{-1}$, via increasing penalty parameter value. (d) Optimal actuator for $\beta = 1$, via increasing penalty parameter value.

For $J_{2,h}$ setting, the results are presented in Figure 4 and Table 4. The advantage of a continuation approach is that the cost $J_{2,h}$ obtained for the final value of $\beta = 1$ is 0.0087 smaller than the one without initialization procedure. So, the optimal actuator size is given by 0.1841.

TABLE 4. The optimization values for $y(x, 0) = \sin^8(3\pi x)$, $v(x, 0) = 0$. Each row, except for the last row with $\beta = 1^*$, is initialized with the final actuator corresponding to the previous one.

| $\beta$ | $J_{2,h}(\omega)$ | $J_{2,h}^{LQ}(\omega)$ | $J_{2,h}^{\beta}(\omega)$ | actuator size | iterations |
|---|---|---|---|---|---|
| $10^{-2}$ | 0.1067 | 0.1066 | $1.459 \times 10^{-4}$ | 0.2697 | 200 |
| $10^{-1}$ | 0.1135 | 0.1134 | $1.175 \times 10^{-4}$ | 0.1802 | 12 |
| 1 | 0.1061 | 0.1053 | $7.632 \times 10^{-4}$ | 0.1841 | 20 |
| $1^*$ | 0.1148 | 0.1134 | $1.400 \times 10^{-3}$ | 0.1785 | 10 |




(a) Initial displacement $y(x,0)$

(b) $J_{1,h}$ cost history

(c) $\beta = 1$, without initialization

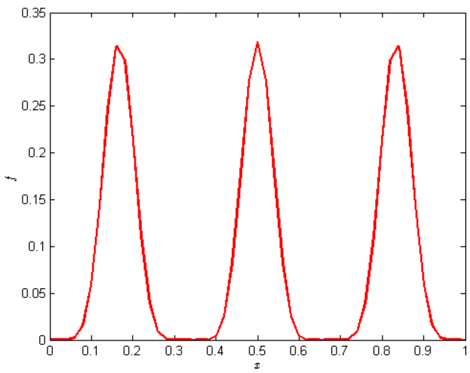
(d) $\beta = 1$

FIGURE 3. (a) Initial displacement $y(x,0) = \sin^8(3\pi x)$. (b) Decay of total cost $J_{1,h}$. (c) Optimal actuator for $\beta = 1$, without initialization via increasing penalty parameter value. (d) Optimal actuator for $\beta = 1$, via increasing penalty parameter value.

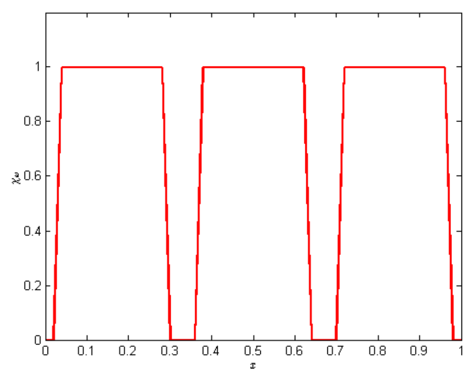
## 5. Conclusion

In this paper, we derived the topological derivatives for determining the optimal actuator design for a linear wave equation using the averaged adjoint approach. Also, we proposed a homotopy continuation algorithm initialized by a topological derivative. For numerical realization, a mixture of weighted finite difference, and finite element methods were used to solve for the state, adjoint, and hence, $J_h$. For the two selected numerical examples, we obtained the optimal actuator design for a linear wave equation. We developed the theory for spatial domain $\mathbb{R}^d, d = 1, 2$ but restricted the implementation to a 1-dimensional domain. The extension of this part to a 2-dimensional domain implementation is under our current study plan. Moreover, we remark that the study can be extended to optimal actuator design using topological derivatives embedded into a level-set method.

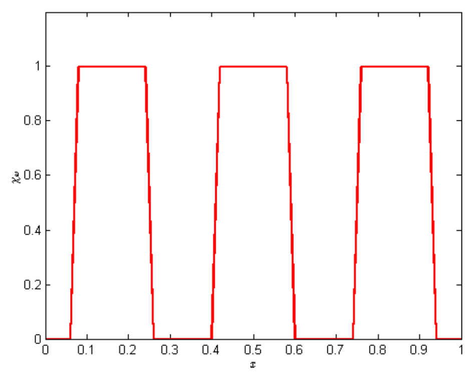
**Acknowledgements.** This study was supported by the SIDA bilateral programme (2015–2022) with Makerere University; Project 316: Capacity building in Mathematics. The authors sincerely thank the anonymous referees for valuable comments and suggestions that led to the final version of the paper.



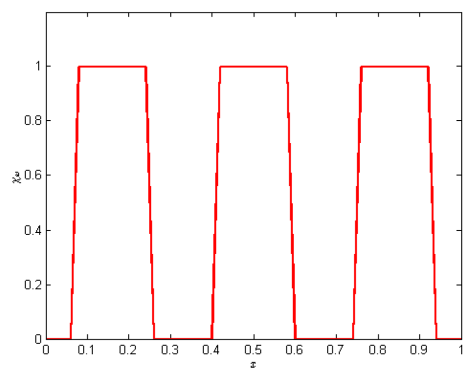

(a) $f$, where $(f, g) \in \mathfrak{X}_2(\omega)$




(b) $\beta = 10^{-2}$, without initialization




(c) $\beta = 1$, without initialization




(d) $\beta = 1$

FIGURE 4. (a) The normalized initial displacement $f$. (b) Final actuator for $\beta = 10^{-2}$, without initialization. (c) Optimal actuator for $\beta = 1$, without initialization. (d) Optimal actuator for $\beta = 1$, via increasing penalty parameter value.

## Conflict of Interest

The authors declare that they have no conflict of interest.

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Corresponding author, Department of Mathematics, Muni University, Arua, Uganda
*Email address*: `d.arop@muni.ac.ug`

Department of Mathematics, Makerere University, Kampala, Uganda
*Email address*: `henry.kasumba@mak.ac.ug`

Department of Mathematics, Makerere University, Kampala, Uganda
*Email address*: `juma.kasozi@mak.ac.ug`

Department of Mathematics, Linköping University, Linköping, Sweden
*Email address*: `fredrik.berntsson@liu.se`